

Expected Utility Consistent Extensions of Preferences

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Abstract

We consider the problem of extending a (complete) order over a set to its power set. The extension axioms we consider generate orderings over sets according to their expected utilities induced by some assignment of utilities over alternatives and probability distributions over sets. The model we propose gives a general and unified exposition of expected utility consistent extensions while it allows to emphasize various subtleties, the effects of which seem to be underestimated - particularly in the literature on strategy-proof social choice correspondences.

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1 Introduction

We consider the problem of extending a (complete) order over a set to its power set. The set under consideration is interpreted as a set of alternatives and we conceive orders as individual preferences. A central concept of this analysis is an *extension axiom* which is a rule that determines how an individual with a given preference over alternatives is required to rank certain sets.¹ Given an extension axiom, the *compatibility* of a preference over sets with a preference over alternatives is the obedience of the extended order to the requirements of the extension axiom.²

The choice of extension axioms depends on the interpretations of sets. We conceive a set as a list of mutually incompatible alternatives, i.e., as a first refinement of the initial set of alternatives from which the final unique choice will be later made.³ As social choice correspondences are typically social choice rules which give non-resolute outcomes, the problem we consider is connected to the analysis of strategy-proof social choice correspondences.⁴

Our focus is on extension axioms that order sets according to their expected utilities induced by some assignment of utilities over alternatives and probability distributions over sets. This approach leads to what is generally called the *expected utility consistent extension of a preference*. Nevertheless, the idea needs to be made more precise by determining which utility functions and probability distributions are admissible. Moreover, whether the order generated by expected utilities is complete or partial also matters. In fact, completing a generated partial order and directly generating a complete order may lead to different admissible orderings. The literature seems to be missing a unified exposition of these subtleties - a treatment of which is one of our aims. So after introducing the basic notation and notions in Section 2, we

¹Technically speaking, an extension axiom assigns to each ordering of alternatives, an antisymmetric and transitive binary relation over sets.

²To be more formal, given an extension axiom π , a complete order R over sets is compatible with an order ρ over alternatives if and only if R is a completion of the partial order $\pi(\rho)$ that π assigns to ρ .

³Of course, there are other interpretations of sets such as being a list of mutually compatible alternatives or a menu from which the individual whose preference under consideration makes a choice. All these interpretations have their own literature and axioms, which we leave outside the scope of this paper. A general account of the literature on extending an order over a set to its power set is given by Barberà, Bossert and Pattanaik (2004).

⁴The literature on strategy-proof social choice correspondences contains Fishburn (1972), Pattanaik (1973), Gärdenfors (1976), Barberà (1977), Kelly (1977), Feldman (1980), Duggan and Schwarz (2000), Barberà, Dutta and Sen (2001), Benoit (2002), Ching and Zhou (2002), Ozyurt and Sanver (2006). This list is certainly non-exhaustive. One can see Taylor (2005) for an excellent account of the literature.

devote Section 3 to give an account of expected utility consistent extensions in our unified framework. In Section 4, we discuss the effects of our findings to definitions of strategy-proofness. Moreover, we are able to remark that not all the finesses of expected utility consistent extensions are incorporated into the literature on strategy-proof social choice correspondences. Section 5 concludes.

2 Basic Notions

Consider a finite non-empty set of alternatives A and let $\underline{A} = 2^A \setminus \{\emptyset\}$. We denote Π for the set of complete, transitive and antisymmetric binary relations over A and \mathfrak{R} for the set of complete and transitive binary relations over \underline{A} . We write $\rho \in \Pi$ and $R \in \mathfrak{R}$ for typical orders over A and \underline{A} , respectively. We let P stand for the strict counterpart of $R \in \mathfrak{R}$.⁵ We write U_ρ for the set of all (real-valued) utility functions over A that represent $\rho \in \Pi$.⁶

By an *extension axiom*, we mean a mapping π which assigns to each $\rho \in \Pi$ a strict partial order⁷ $\pi(\rho)$ of \underline{A} such that for all distinct $x, y \in A$ we have $x \rho y \iff \{x\} \pi(\rho) \{y\}$. Given any extension axiom π and any $\rho \in \Pi$, we write $D^\pi(\rho) = \{R \in \mathfrak{R} : X \pi(\rho) Y \Rightarrow X P Y \text{ for all distinct } X, Y \in \underline{A}\}$ for the set of complete and transitive binary relations over \underline{A} which are compatible with $\pi(\rho)$.⁸

Let Ω_X be the set of all non-degenerate probability distributions over $X \in \underline{A}$, i.e., each $\omega_X \in \Omega_X$ is a probability distribution $\{\omega_X(x)\}_{x \in X}$ over X where $\omega_X(x) \in (0, 1]$ is interpreted as the (positive) probability that $x \in X$ will be chosen from X .⁹ We call $\Omega = \prod_{X \in \underline{A}} \Omega_X$ the set of *priors* (over \underline{A}). So a prior $\omega = (\omega_X)_{X \in \underline{A}} \in \Omega$ is a vector which collects a probability distribution over each element of \underline{A} . Any given non-empty set $\Gamma \subseteq \Omega$ of admissible priors over \underline{A} induces an extension axiom π^Γ which assigns to each $\rho \in \Pi$ a binary relation $\pi^\Gamma(\rho)$ over \underline{A} as follows: For all distinct $X, Y \in \underline{A}$, we have $X \pi^\Gamma(\rho) Y$ if and only if $\sum_{x \in X} \omega_X(x) \cdot u(x) > \sum_{y \in Y} \omega_Y(y) \cdot u(y) \forall u \in U_\rho, \forall \omega \in \Gamma$.¹⁰ So $D^{\pi^\Gamma}(\rho)$ is the set of orderings which are completions of the partial

⁵So for any $X, Y \in \underline{A}$, we have $X P Y$ whenever $X R Y$ holds but $Y R X$ does not.

⁶A utility function u over A represents $\rho \in \Pi$ iff $u(x) \geq u(y) \iff x \rho y \forall x, y \in A$.

⁷A strict partial order is a transitive and antisymmetric (but not necessarily complete) binary relation.

⁸So every $R \in D^\pi(\rho)$ is a completion of the strict partial order $\pi(\rho)$ and $D^\pi(\rho)$ is non-empty by Spilrajn's Theorem.

⁹So we have $\sum_{x \in X} \omega_X(x) = 1$ for all $X \in \underline{A}$.

¹⁰One can immediately check that π^Γ is an extension axiom, i.e., $\pi^\Gamma(\rho)$ is transitive and

order π^Γ that the set of admissible priors Γ induces. We call $D^{\pi^\Gamma}(\rho)$ the set of orderings over \underline{A} which are *expected utility consistent with ρ* (under the set of admissible priors Γ). Note that for any $\rho \in \Pi$ and any $R \in \mathfrak{R}$ we have $R \in D^{\pi^\Gamma}(\rho) \iff \forall X, Y \in \underline{A}$ with $X R Y$, there exists $(u, \omega) \in U_\rho \times \Gamma$ such that $\sum_{x \in X} \omega_X(x).u(x) \geq \sum_{y \in Y} \omega_Y(y).u(y)$. One could impose a stronger expected utility consistency requirement by reversing the order of the quantifiers. In other words, one could say that $R \in \mathfrak{R}$ is *strongly expected utility consistent with $\rho \in \Pi$* (under the set of admissible priors Γ) iff there exists $(u, \omega) \in U_\rho \times \Gamma$ such that $X R Y \iff \sum_{x \in X} \omega_X(x).u(x) \geq \sum_{y \in Y} \omega_Y(y).u(y)$ for all $X, Y \in \underline{A}$. We write $D^\Gamma(\rho)$ for the set of orderings over \underline{A} which are *strongly expected utility consistent with $\rho \in \Pi$* . In what follows, we say that a triple $(\rho, u, \omega) \in \Pi \times U_\rho \times \Gamma$ *directly generates* $R \in \mathfrak{R}$ iff $X R Y \iff \sum_{x \in X} \omega_X(x).u(x) \geq \sum_{y \in Y} \omega_Y(y).u(y)$ for all $X, Y \in \underline{A}$. So $D^\Gamma(\rho)$ is the set of orderings over \underline{A} which are directly generated by some $(\rho, u, \omega) \in \Pi \times U_\rho \times \Gamma$. Note that $D^\Gamma(\rho) \subseteq D^{\pi^\Gamma}(\rho) \forall \rho \in \Pi$ follows from the definitions. On the other hand, as we show in Section 3.2, the properness of the set inclusion depends on the choice of admissible priors Γ .

3 An Account of Expected Utility Consistent Extensions

3.1 The choice of admissible priors

The precise meaning of the “expected utility consistency” of an extension depends on the set of admissible priors and the set of admissible utility functions. Given a preference $\rho \in \Pi$ over alternatives, we let any $u \in U_\rho$ to be admissible. On the other hand, we allow the set of admissible priors Γ to vary. The literature exhibits three choices of Γ :

3.1.1 General Expected Utility Consistency (GEUC)

Any prior is allowed, i.e., $\Gamma = \Omega$. As one can also deduce from Theorem 4.4.1 in Taylor (2005), the extension axiom π^Ω induced by GEUC is equivalent to the following extension axiom introduced by Kelly (1977):

The Kelly Principle: For each $\rho \in \Pi$, let $\pi^{KELLY}(\rho) = \{(X, Y) \in \underline{A} \times \underline{A} \setminus \{X\} : x \rho y \forall x \in X \forall y \in Y\}$.

antisymmetric, while $x \rho y \iff \{x\} \pi^\Gamma(\rho) \{y\}$ for all distinct $x, y \in A$.

Theorem 3.1 $\pi^\Omega(\rho) = \pi^{KELLY}(\rho) \forall \rho \in \Pi$.

Proof. Take any $\rho \in \Pi$. To see $\pi^{KELLY}(\rho) \subseteq \pi^\Omega(\rho)$, pick some $(X, Y) \in \pi^{KELLY}(\rho)$. Let $x_o \in X$ be such that $x \rho x_o \forall x \in X$ and $y_0 \in Y$ be such that $y_0 \rho y \forall y \in Y$. As $(X, Y) \in \pi^{KELLY}(\rho)$ we have $x_o \rho y_0$. Thus, for any $u \in U_\rho$, any $\omega_X \in \Omega_X$ and any $\omega_Y \in \Omega_Y$, we have $\sum_{x \in X} \omega_x(x).u(x) \geq u(x_o) \geq u(y_0) \geq \sum_{y \in Y} \omega_y(y).u(y)$. If $X \cap Y = \emptyset$, then $u(x_o) > u(y_0)$, implying $\sum_{x \in X} \omega_x(x).u(x) > \sum_{y \in Y} \omega_y(y).u(y)$. If $X \cap Y \neq \emptyset$, then at least one of X and Y is not a singleton as otherwise X and Y would coincide. In case X is not a singleton we have $\sum_{x \in X} \omega_x(x).u(x) > u(x_o)$ and in case Y is not a singleton we have $u(y_0) > \sum_{y \in Y} \omega_y(y).u(y)$, both of which implies $\sum_{x \in X} \omega_x(x).u(x) > \sum_{y \in Y} \omega_y(y).u(y)$, showing that $(X, Y) \in \pi^\Omega(\rho)$.

To see $\pi^\Omega(\rho) \subseteq \pi^{KELLY}(\rho)$, pick some $(X, Y) \notin \pi^{KELLY}(\rho)$. So there exist $y_0 \in Y$ and $x_o \in X \setminus \{y_0\}$ with $y_0 \rho x_o$. Now, let $x_1 \in X$ be such that $x_1 \rho x \forall x \in X$. Take any $u \in U_\rho$ and any $r \in (0, 1)$ which satisfies $r.u(x_1) + (1-r).[u(x_o) - u(y_0)] < 0$. So $r.u(x_1) < (1-r).[u(y_0) - u(x_o)]$. Let $\omega_X(x_o) = \omega_Y(y_o) = 1 - r$. So we have

$$\begin{aligned} \sum_{x \in X} \omega_x(x).u(x) &\leq \omega_X(x_o).u(x_o) + (1 - \omega_X(x_o)).u(x_1) \\ &= (1 - r).u(x_o) + r.u(x_1) \\ &< (1 - r).u(x_o) + (1 - r).[u(y_0) - u(x_o)] \\ &= (1 - r).u(y_0) = \omega_Y(y_o).u(y_0) \\ &\leq \sum_{y \in Y} \omega_y(y).u(y) \end{aligned}$$

which implies $(X, Y) \notin \pi^\Omega(\rho)$. ■

3.1.2 Bayesian Expected Utility Consistency (BEUC)

This is a restriction of GEUC that Barberà, Dutta and Sen (2001) and Ching and Zhou (2002) use in their analysis of strategy-proof social choice correspondences.¹¹ The set of admissible priors is defined as $\Gamma^{BEUC} = \{\omega \in \Omega : \omega_X(x) = \frac{\omega_A(x)}{\sum_{y \in X} \omega_A(y)} \text{ for all } X \in \underline{A} \setminus \{A\} \text{ and for all } x \in X\}$. As one can also deduce from Lemma 1 of Ching and Zhou (2002), the extension axiom $\pi^{\Gamma^{BEUC}}$ induced by BEUC is equivalent to the following extension axiom introduced by Gärdenfors (1976):

¹¹Barberà, Dutta and Sen (2001) call it Conditional Expected Utility Consistency.

The Gärdenfors Principle: For each $\rho \in \Pi$, let $\pi^{GF}(\rho) = \{(X, Y) \in \underline{A} \times \underline{A} \setminus \{X\} : (x \rho y \forall x \in X \setminus Y \forall y \in Y) \text{ and } (x \rho y \forall x \in X \forall y \in Y \setminus X)\}$.

The proof of the equivalence theorem we will state benefits from the following two lemmata which we prove in Appendix A.

Lemma 3.1 For all $\rho \in \Pi$ and all $(X, Y) \in \pi^{GF}(\rho)$ with $X \cap Y \neq \emptyset$ and $X \setminus Y \neq \emptyset$, we have $(X, X \cap Y) \in \pi^{\Gamma^{BEUC}}(\rho)$.

Lemma 3.2 For all $\rho \in \Pi$ and all $(X, Y) \in \pi^{GF}(\rho)$ with $X \cap Y \neq \emptyset$ and $Y \setminus X \neq \emptyset$ we have $(X \cap Y, Y) \in \pi^{\Gamma^{BEUC}}(\rho)$.

Theorem 3.2 $\pi^{\Gamma^{BEUC}}(\rho) = \pi^{GF}(\rho) \forall \rho \in \Pi$.

Proof. Take any $\rho \in \Pi$. We first show $\pi^{GF}(\rho) \subseteq \pi^{\Gamma^{BEUC}}(\rho)$. Take any $(X, Y) \in \pi^{GF}(\rho)$. Consider the following 4 exhaustive cases:

CASE 1: $X \cap Y \neq \emptyset, X \setminus Y \neq \emptyset, Y \setminus X = \emptyset$. So $Y = (X \cap Y) \subset X$ and by Lemma 3.1, we have $(X, X \cap Y) \in \pi^{\Gamma^{BEUC}}(\rho)$, thus $(X, Y) \in \pi^{\Gamma^{BEUC}}(\rho)$.

CASE 2: $X \cap Y \neq \emptyset, Y \setminus X \neq \emptyset, X \setminus Y = \emptyset$. So $X = (X \cap Y) \subset Y$ and by Lemma 3.2, we have $(X \cap Y, Y) \in \pi^{\Gamma^{BEUC}}(\rho)$, thus $(X, Y) \in \pi^{\Gamma^{BEUC}}(\rho)$.

CASE 3: $X \cap Y \neq \emptyset, Y \setminus X \neq \emptyset, X \setminus Y \neq \emptyset$. The conjunction of Lemma 3.1 and Lemma 3.2 implies $(X, X \cap Y) \in \pi^{\Gamma^{BEUC}}(\rho)$ and $(X \cap Y, Y) \in \pi^{\Gamma^{BEUC}}(\rho)$ while by transitivity we have $(X, Y) \in \pi^{\Gamma^{BEUC}}(\rho)$.

CASE 4: $X \cap Y = \emptyset$. As $(X, Y) \in \pi^{GF}(\rho)$, we have $x \rho y \forall x \in X, \forall y \in Y$. So $\sum_{x \in X} \omega_X(x)u(x) > \sum_{y \in Y} \omega_Y(y)u(y)$ holds for all $u \in U_\rho$ and all $\omega \in \Gamma^{BEUC}$, showing $(X, Y) \in \pi^{\Gamma^{BEUC}}(\rho)$.

We now show $\pi^{\Gamma^{BEUC}}(\rho) \subseteq \pi^{GF}(\rho)$. Take some $(X, Y) \in \underline{A} \times \underline{A} \setminus \{X\}$ with $(X, Y) \notin \pi^{GF}(\rho)$. So at least one of the following two conditions holds:

(i) $\exists x \in X \setminus Y, \exists y \in Y$ such that $y \rho x$

(ii) $\exists x \in X, \exists y \in Y \setminus X$ such that $y \rho x$

First let (i) hold. Let $a \in X \setminus Y$ be such that $x \rho a \forall x \in X \setminus Y$ and $b \in Y$ be such that $b \rho y \forall y \in Y$. As (i) holds, we have $b \rho a$. Now fix some $u \in U_\rho$. Take some $\epsilon \in (0, 1)$ and consider the prior $\omega \in \Gamma^{BEUC}$ where $\omega_A(a) = \omega_A(b) = \frac{1-\epsilon}{2}$ and $\omega_A(x) = \frac{\epsilon}{\#A-2} \forall x \in A \setminus \{a, b\}$. Consider first the case where $b \in X$. We have

$$\sum_{x \in X} \omega_X(x)u(x) = \frac{1}{\sum_{x \in X} \omega_A(x)} \left(\frac{1-\epsilon}{2}u(a) + \frac{1-\epsilon}{2}u(b) + (\#X-2) \frac{\epsilon}{\#A-2} \sum_{x \in X \setminus \{a, b\}} u(x) \right)$$

and $\sum_{y \in Y} \omega_Y(y)u(y) = \frac{1}{\sum_{y \in Y} \omega_A(y)} \left(\frac{1-\epsilon}{2}u(b) + (\#Y - 1) \frac{\epsilon}{\#A-2} \sum_{y \in Y \setminus \{b\}} u(y) \right)$. So when ϵ is picked arbitrarily small, $\sum_{x \in X} \omega_X(x)u(x)$ approaches to $\frac{u(a)+u(b)}{2}$ while $\sum_{y \in Y} \omega_Y(y)u(y)$ approaches to $u(b)$ and as $u(b) > u(a)$, this allows $\sum_{y \in Y} \omega_Y(y)u(y) > \sum_{x \in X} \omega_X(x)u(x)$, showing that $(X, Y) \notin \pi^{\Gamma^{BEUC}}(\rho)$. Now consider the case where $b \notin X$. We have $\sum_{x \in X} \omega_X(x)u(x) = \frac{1}{\sum_{x \in X} \omega_A(x)} \left(\frac{1-\epsilon}{2}u(a) + (\#X - 1) \frac{\epsilon}{\#A-2} \sum_{x \in X \setminus \{a, b\}} u(x) \right)$ and $\sum_{y \in Y} \omega_Y(y)u(y) = \frac{1}{\sum_{y \in Y} \omega_A(y)} \left(\frac{1-\epsilon}{2}u(b) + (\#Y - 1) \frac{\epsilon}{\#A-2} \sum_{y \in Y \setminus \{b\}} u(y) \right)$. So when ϵ is picked arbitrarily small, $\sum_{x \in X} \omega_X(x)u(x)$ approaches to $u(a)$ while $\sum_{y \in Y} \omega_Y(y)u(y)$ approaches to $u(b)$ and as $u(b) > u(a)$, this allows $\sum_{y \in Y} \omega_Y(y)u(y) > \sum_{x \in X} \omega_X(x)u(x)$, showing $(X, Y) \notin \pi^{\Gamma^{BEUC}}(\rho)$. Now let (ii) hold. Let $a \in X$ be such that $x \rho a \forall x \in X$ and $b \in Y \setminus X$ be such that $b \rho y \forall y \in Y \setminus X$. As (ii) holds, we have $b \rho a$. Fixing some $u \in U_\rho$, taking some $\epsilon \in (0, 1)$ and considering a prior $\omega \in \Gamma^{BEUC}$ as above, one can obtain $\sum_{y \in Y} \omega_Y(y)u(y) > \sum_{x \in X} \omega_X(x)u(x)$, showing $(X, Y) \notin \pi^{\Gamma^{BEUC}}(\rho)$. ■

3.1.3 Equal-Probability Expected Utility Consistency (EEUC)

This is a restriction of BEUC (hence of GEUC) that Feldman (1980) and Barberà, Dutta and Sen (2001) use in their analysis of strategy-proof social choice correspondences.¹² Letting ω^\approx be defined for each $X \in \underline{A}$ as $\omega^\approx_X(x) = \frac{1}{\#X}$ for all $x \in X$, we have $\Gamma^{EEUC} = \{\omega^\approx\}$. We characterize Γ^{EEUC} in terms of an axiom that we call *componentwise dominance*. We define two equivalent versions of it.

The Componentwise Dominance Principle 1: For any real number r , we write $\lceil r \rceil$ for the lowest integer no less than r . Let N stand for the set of natural numbers. Picking any two $m, n \in N$, we introduce a mapping

¹²Barberà, Dutta and Sen (2001) call it Conditional Expected Utility Consistency With Equal Probabilities.

$f_{mn} : N \longrightarrow N$ defined for each $i \in N$ as $f_{mn}(i) = \lceil \frac{1+n \cdot (i-1)}{m} \rceil$. Note that f_{mn} is an increasing function on N . Now take any $\rho \in \Pi$ and any distinct $X, Y \in \underline{A}$. Let, without loss of generality, $X = \{x_1, \dots, x_{\#X}\}$ with $x_i \rho x_{i+1} \forall i \in \{1, \dots, \#X - 1\}$ and $Y = \{y_1, \dots, y_{\#Y}\}$ with $y_j \rho y_{j+1} \forall j \in \{1, \dots, \#Y - 1\}$. The *componentwise dominance principle 1* is defined through the strict partial order $\pi^{CD1}(\rho) = \{(X, Y) \in \underline{A} \times \underline{A} \setminus \{X\} : x_i \rho y_{f_{\#X \#Y}(i)} \forall i \in \{1, \dots, \#X\}\}$.¹³

The Componentwise Dominance Principle 2: Take any $\rho \in \Pi$ and any $X = \{x_1, \dots, x_{\#X}\} \in \underline{A}$ with $x_i \rho x_{i+1} \forall i \in \{1, \dots, \#X - 1\}$. Given any $t \in N$, we define a $t \cdot \#X$ dimensional vector \vec{X}^t such that given any $i \in \{1, \dots, t \cdot \#X\}$, we have $\vec{X}_i^t = x_{\lceil \frac{i}{\#X} \rceil}$.¹⁴ In other words, we can write $\vec{X}^t = (x_1, \dots, x_1, \dots, x_{\#X}, \dots, x_{\#X})$ where each $x \in X$ appears t times while given any $x_i, x_j \in X$ with $i < j$, x_i appears at the left of x_j . Take also $Y = \{y_1, \dots, y_{\#Y}\} \in \underline{A} \setminus \{X\}$ with $y_i \rho y_{i+1} \forall i \in \{1, \dots, \#Y - 1\}$ and define \vec{Y}^t similarly. The *componentwise dominance principle 2* is defined through the strict partial order $\pi^{CD2}(\rho) = \{(X, Y) \in \underline{A} \times \underline{A} \setminus \{X\} : \vec{X}_i^{\#Y} \rho \vec{Y}_i^{\#X} \forall i \in \{1, \dots, \#X \cdot \#Y\}\}$.¹⁵

Lemma 3.3 For all $\rho \in \Pi$, we have $\pi^{CD1}(\rho) = \pi^{CD2}(\rho)$.

Proof. Take any $\rho \in \Pi$. To see $\pi^{CD1}(\rho) \subseteq \pi^{CD2}(\rho)$, pick some $(X, Y) \in \pi^{CD1}(\rho)$. Now take any $k \in \{1, \dots, \#X \cdot \#Y\}$. We have $\vec{X}_k^{\#Y} = x_{\lceil \frac{k}{\#Y} \rceil}$ and $\vec{Y}_k^{\#X} = y_{\lceil \frac{k}{\#X} \rceil}$. As $(X, Y) \in \pi^{CD1}(\rho)$, we have $x_{\lceil \frac{k}{\#Y} \rceil} \rho y_{f_{\#X \#Y}(\lceil \frac{k}{\#Y} \rceil)}$. Now check that $f_{\#X \#Y}(\lceil \frac{k}{\#Y} \rceil) \leq \lceil \frac{k}{\#X} \rceil$ for all $k \in \{1, \dots, \#X \cdot \#Y\}$. As a result, $y_{f_{\#X \#Y}(\lceil \frac{k}{\#Y} \rceil)} \rho y_{\lceil \frac{k}{\#X} \rceil}$, which implies $x_{\lceil \frac{k}{\#Y} \rceil} \rho y_{\lceil \frac{k}{\#X} \rceil}$, showing that $(X, Y) \in \pi^{CD2}(\rho)$.

To see $\pi^{CD2}(\rho) \subseteq \pi^{CD1}(\rho)$, pick some $(X, Y) \in \pi^{CD2}(\rho)$. So $\vec{X}_i^{\#Y} \rho \vec{Y}_i^{\#X} \forall i \in \{1, \dots, \#X \cdot \#Y\}$. Suppose, for a contradiction, that $(X, Y) \notin \pi^{CD1}(\rho)$. So there exists $i \in \{1, \dots, \#X\}$ such that $x_i \rho y_{f_{\#X \#Y}(i)}$ fails. Thus, if $x_i \rho y_j$ for some $y_j \in Y$ then $j \geq f_{\#X \#Y}(i) + 1$. This, combined with the fact that $\vec{X}_i^{\#Y} \rho \vec{Y}_i^{\#X}$ for each $i \in \{1, \dots, \#X \cdot \#Y\}$, implies $(i - 1) \cdot \#Y \geq f_{\#X \#Y}(i) \cdot \#X$, which in turn implies $f_{\#X \#Y}(i) \leq (i - 1) \cdot \frac{\#Y}{\#X}$, contradicting the definition of $f_{\#X \#Y}$, hence showing $\pi^{CD2}(\rho) \subseteq \pi^{CD1}(\rho)$. ■

So, for each $\rho \in \Pi$, we write $\pi^{CD}(\rho) = \pi^{CD1}(\rho) = \pi^{CD2}(\rho)$.

Theorem 3.3 $\pi^{CD}(\rho) = \pi^{\Gamma^{EEUC}}(\rho) \forall \rho \in \Pi$.

¹³The fact that $\pi^{CD1}(\rho)$ is a strict partial order may not be visible at the first glance and we discuss the matter at the end of the section.

¹⁴As usual, \vec{X}_i^t is the i^{th} entry of \vec{X}^t .

¹⁵The fact that $\pi^{CD2}(\rho)$ is a strict partial order may not be visible at the first glance and we discuss the matter at the end of the section.

Proof. Take any $\rho \in \Pi$. To see $\pi^{CD}(\rho) \subseteq \pi^{\Gamma^{EEUC}}(\rho)$, pick some $(X, Y) \in \pi^{CD}(\rho)$. So $\vec{X}_i^{\#Y} \rho \vec{Y}_i^{\#X} \forall i \in \{1, \dots, \#X.\#Y\}$. Thus, for any $u \in U_\rho$, we have $\sum_{i=1}^{\#X.\#Y} u(\vec{X}_i^{\#Y}) > \sum_{i=1}^{\#X.\#Y} u(\vec{Y}_i^{\#X})$, the inequality being strict due to the fact that X and Y are distinct. This inequality can be rewritten as $\sum_{i=1}^{\#X} \#Y.u(x_i) > \sum_{j=1}^{\#Y} \#X.u(y_j)$, which implies $\frac{\sum_{i=1}^{\#X} u(x_i)}{\#X} > \frac{\sum_{j=1}^{\#Y} u(y_j)}{\#Y}$, thus showing $(X, Y) \in \Pi^{\Gamma^{EEUC}}(\rho)$.

To see $\Pi^{\Gamma^{EEUC}}(\rho) \subseteq \Pi^{CD}(\rho)$, pick some $(X, Y) \notin \Pi^{CD}(\rho)$. So there exists $j \in \{1, \dots, \#X\}$ such that $x_j \rho y_{f_{\#X} \#Y(j)}$ fails, hence $u(x_j) < u(y_{f_{\#X} \#Y(j)})$ for any $u \in U_\rho$. Now, let $X \cup Y = Z = \{z_1, \dots, z_{\#Z}\}$ with $z_i \rho z_{i+1} \forall i \in \{1, \dots, \#Z - 1\}$ and take some $\epsilon > 0$ and some $M > 0$. Let $z_k \in Z$ coincide with x_j . Consider the following $u \in U_\rho$ defined as $u(z_{\#Z}) = 0$, $u(z_i) - u(z_{i+1}) = \epsilon$ for all $i \in \{k, \dots, \#Z - 1\}$, $u(z_{k-1}) - u(z_k) = M$, and $u(z_i) - u(z_{i+1}) = \epsilon$ for all $i \in \{1, \dots, k - 2\}$. Picking M arbitrarily

large and ϵ arbitrarily close to 0, we have $\frac{\sum_{j=1}^{\#Y} u(y_j)}{\#Y} > \frac{\sum_{i=1}^{\#X} u(x_i)}{\#X}$, showing that $(X, Y) \notin \Pi^{\Gamma^{EEUC}}(\rho)$. ■

We close by noting the straightforwardness of checking that $\pi^{\Gamma^{EEUC}}(\rho)$ is a strict partial order, thus answering the issue raised by Footnotes 13 and 15.

3.2 Completing partial orders versus direct generation of complete orderings

Whether an ordering over sets is obtained by completing a partial order generated through expected utilities (i.e., expected utility consistency) or is directly generated with reference to expected utilities (i.e., strong expected utility consistency) matters. In other words, given a set Γ of admissible priors, the extension axiom π^Γ induced by Γ and a preference $\rho \in \Pi$, the sets $D^\Gamma(\rho)$ and $D^{\pi^\Gamma}(\rho)$ need not coincide. In fact, as we note in Section 2, $D^\Gamma(\rho)$ being a subset of $D^{\pi^\Gamma}(\rho)$ follows from the definitions. A formal statement of this logical relationship is given by the following theorem.

Theorem 3.4 *Given any set Γ of admissible priors over \underline{A} , we have $D^\Gamma(\rho) \subseteq D^{\pi^\Gamma}(\rho) \forall \rho \in \Pi$.*

Proof. Take any set Γ of admissible priors over \underline{A} , any $\rho \in \Pi$ and any

$R^* \in \mathfrak{R} \setminus D^{\pi^\Gamma}(\rho)$. So there exist distinct $X, Y \in \underline{A}$ with $Y R^* X$ while $\sum_{x \in X} \omega_X(x).u(x) > \sum_{y \in Y} \omega_Y(y).u(y) \forall u \in U_\rho, \forall \omega \in \Gamma$. Thus, there exists no $(\rho, u, \omega) \in \Pi \times U_\rho \times \Gamma$ that directly generates R^* , showing $R^* \notin D^\Gamma(\rho)$. ■

Whether the set inclusion announced by Theorem 3.4 is proper or not depends on the choice of admissible priors Γ . To explore this, we define the (strong) leximax extension $\lambda^+(\rho) \in \mathfrak{R}$ and the (strong) leximin extension $\lambda^-(\rho) \in \mathfrak{R}$ of $\rho \in \Pi$.¹⁶ Under the leximax extension, sets are ordered according to their best elements. If these are the same, then the ordering is made according to the second best elements, etc. The elements according to which the sets are compared will disagree at some step – except possibly when one set is a subset of the other, in which case the smaller set is preferred.¹⁷ To speak formally, given any $\rho \in \Pi$, the *leximax extension* $\lambda^+(\rho) \in \mathfrak{R}$ is defined as follows: Take any distinct $X, Y \in \underline{A}$. First consider the case where $\#X = \#Y = k$ for some $k \in \{1, \dots, \#A - 1\}$. Let, without loss of generality, $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_k\}$ such that $x_j \rho x_{j+1}$ and $y_j \rho y_{j+1}$ for all $j \in \{1, \dots, k - 1\}$. We have $X \lambda^+(\rho) Y$ if and only if $x_h \rho y_h$ for the smallest $h \in \{1, \dots, k\}$ such that $x_h \neq y_h$. Now consider the case where $\#X \neq \#Y$. Let, without loss of generality, $X = \{x_1, \dots, x_{\#X}\}$ and $Y = \{y_1, \dots, y_{\#Y}\}$ such that $x_j \rho x_{j+1}$ for all $j \in \{1, \dots, \#X - 1\}$ and $y_j \rho y_{j+1}$ for all $j \in \{1, \dots, \#Y - 1\}$. We have either $x_h = y_h$ for all $h \in \{1, \dots, \min\{\#X, \#Y\}\}$ or there exists some $h \in \{1, \dots, \min\{\#X, \#Y\}\}$ for which $x_h \neq y_h$. For the first case, $X \lambda^+(\rho) Y$ if and only if $\#X < \#Y$. For the second case, $X \lambda^+(\rho) Y$ if and only if $x_h \rho y_h$ for the smallest $h \in \{1, \dots, \min\{\#X, \#Y\}\}$ such that $x_h \neq y_h$.

The concept of a leximin extension is similarly defined while it is based on ordering two sets according to a lexicographic comparison of their worst elements. Again the elements according to which the sets are compared will disagree at some step – except possibly when one set is a subset of the other, in which case the larger set is preferred.¹⁸ So given any $\rho \in \Pi$, the *leximin extension* $\lambda^-(\rho) \in \mathfrak{R}$ is defined as follows: Take any distinct $X, Y \in \underline{A}$. First consider the case where $\#X = \#Y = k$ for some $k \in \{1, \dots, \#A - 1\}$. Let, without loss of generality, $X = \{x_1, \dots, x_k\}$ and $Y = \{y_1, \dots, y_k\}$ such that $x_j \rho x_{j+1}$ and $y_j \rho y_{j+1}$ for all $j \in \{1, \dots, k - 1\}$.

¹⁶Kaymak and Sanver (2003) show that at each $\rho \in \Pi$, the leximax and leximin extensions determine unique orderings $\lambda^+(\rho)$ and $\lambda^-(\rho)$ over \underline{A} which are complete, transitive and antisymmetric.

¹⁷This is exactly how words are ordered in a dictionary. For example, given three alternatives a, b and c , the leximax extension of the ordering $a \rho b \rho c$ is $\{a\} \lambda^+(\rho) \{a, b\} \lambda^+(\rho) \{a, b, c\} \lambda^+(\rho) \{a, c\} \lambda^+(\rho) \{b\} \lambda^+(\rho) \{b, c\} \lambda^+(\rho) \{c\}$.

¹⁸For example, the leximin extension of the ordering $a \rho b \rho c$ is $\{a\} \lambda^-(\rho) \{a, b\} \lambda^-(\rho) \{b\} \lambda^-(\rho) \{a, c\} \lambda^-(\rho) \{a, b, c\} \lambda^-(\rho) \{b, c\} \lambda^-(\rho) \{c\}$.

We have $X \lambda^-(\rho) Y$ if and only if $x_h \rho y_h$ for the greatest $h \in \{1, \dots, k\}$ such that $x_h \neq y_h$. Now consider the case where $\#X \neq \#Y$. Let, without loss of generality, $X = \{x_1, \dots, x_{\#X}\}$ and $Y = \{y_1, \dots, y_{\#Y}\}$ such that $x_j \rho x_{j+1}$ for all $j \in \{1, \dots, \#X - 1\}$ and $y_j \rho y_{j+1}$ for all $j \in \{1, \dots, \#Y - 1\}$. We have either $x_h = y_h$ for all $h \in \{1, \dots, \min\{\#X, \#Y\}\}$ or there exists some $h \in \{1, \dots, \min\{\#X, \#Y\}\}$ for which $x_h \neq y_h$. For the first case, $X \lambda^-(\rho) Y$ if and only if $\#X > \#Y$. For the second case, $X \lambda^-(\rho) Y$ if and only if $x_h \rho y_h$ for the smallest $h \in \{1, \dots, \min\{\#X, \#Y\}\}$ such that $x_h \neq y_h$.

The first application of Theorem 3.4 is for GEUC, when Ω is taken as the set of admissible priors. In this case, Theorem 3.4 holds as an equality. Before establishing this, we state a lemma.

Lemma 3.4 *Take any one-to-one and real-valued function u defined over A and any $X \in \underline{A}$ with $\#X > 1$. Given any real number $r \in (\min_{x \in X} u(x), \max_{x \in X} u(x))$, there exists $w_X \in \Omega_X$ such that $\sum_{x \in X} w_X(x).u(x) = r$.*

Proof. Let u , X and r be as in the statement of the lemma. Let $x^+, x^- \in X$ be such that $x^+ \rho x \forall x \in X$ and $x \rho x^- \forall x \in X$. We define $X^+ = \{x \in X : u(x) \geq r\}$ and $X^- = \{x \in X : u(x) < r\}$. Both X^+ and X^- are non-empty, as $x^+ \in X^+$ and $x^- \in X^-$. Take any $\omega_{X^+} \in \Omega_{X^+}$ and any $\omega_{X^-} \in \Omega_{X^-}$. Let $q^+ = \sum_{x \in X^+} \omega_{X^+}(x).u(x)$ and $q^- = \sum_{x \in X^-} \omega_{X^-}(x).u(x)$. Note that $q^- < r < q^+$. Let $\lambda = \frac{q^+ - r}{q^+ - q^-} \in (0, 1)$. Now define the following function ω_X over X : For each $x \in X$, we have $\omega_X(x) = (1 - \lambda)\omega_{X^+}(x)$ if $x \in X^+$ and $\omega_X(x) = \lambda\omega_{X^-}(x)$ if $x \in X^-$. It is clear that $\omega_X(x) \in (0, 1)$ for all $x \in X$. Moreover, $\sum_{x \in X} \omega_X(x) = (1 - \lambda) \sum_{x \in X^+} \omega_{X^+}(x) + \lambda \sum_{x \in X^-} \omega_{X^-}(x) = (1 - \lambda) + \lambda = 1$. Thus $\omega_X \in \Omega_X$. Finally, $\sum_{x \in X} \omega_X(x).u(x) = (1 - \lambda) \sum_{x \in X^+} \omega_{X^+}(x).u(x) + \lambda \sum_{x \in X^-} \omega_{X^-}(x).u(x) = (1 - \lambda).q^+ + \lambda q^-$ which, by the choice of λ , equals to r . ■

Theorem 3.5 $D^\Omega(\rho) = D^{\pi^\Omega}(\rho) \forall \rho \in \Pi$.

Proof. Take any $\rho \in \Pi$. The inclusion $D^\Omega(\rho) \subseteq D^{\pi^\Omega}(\rho)$ follows from Theorem 3.4. We now show $D^{\pi^\Omega}(\rho) \subseteq D^\Omega(\rho)$ or by Theorem 3.1 equivalently $D^{\pi^{KELLY}}(\rho) \subseteq D^\Omega(\rho)$. Let $A = \{a_1, \dots, a_m\}$ for some integer $m \geq 2$ and assume, without loss of generality, that $a_i \rho a_{i+1}$ for each $i \in \{1, \dots, m\}$. Take any $R \in D^{\pi^{KELLY}}(\rho)$. Let $C_1 = \{X \in \underline{A} : X R Y \forall Y \in \underline{A}\}$ and define recursively $C_i = \{X \in \underline{A} : X R Y \forall Y \in \underline{A} \setminus \bigcup_{j=1}^{i-1} C_j\}$. So we express R in

terms of a family $\{C_1, \dots, C_k\}$ of equivalence classes where k is some integer that cannot exceed $2^m - 1$. Note that for all $X, Y \in \underline{A}$, we have $X R Y$ if and only if given any $X \in C_i$ and $Y \in C_j$ for some $i, j \in \{1, \dots, k\}$ with $i < j$. As $R \in D^{\pi^{KELLY}}(\rho)$, $C_1 = \{\{a_1\}\}$ and $C_k = \{\{a_m\}\}$. Consider the function $f : \{1, \dots, m\} \rightarrow \{1, \dots, k\}$ where for each $i \in \{1, \dots, m\}$ we have $\{a_i\} \in C_{f(i)}$. So $f(1) = 1$ and $f(m) = k$. Moreover, as $R \in D^{\pi^{KELLY}}(\rho)$, for any $i, j \in \{1, \dots, m\}$ with $i < j$, we have $f(i) < f(j)$. Now we define a real valued utility function u over A as $u(a_i) = k - f(i) + 1$ for each $i \in \{1, \dots, m\}$. We complete the proof by showing the existence of some $\{\omega_X\}_{X \in \underline{A}} \in \Omega$ such that for each $j \in \{1, \dots, k\}$ and for each $X \in C_j$ we have $\sum_{x \in X} \omega_X(x).u(x) = k - j + 1$, as

this ensures that the triple $(\rho, u, \{\omega_X\}_{X \in \underline{A}})$ directly generates R . So take any $j \in \{1, \dots, k\}$ and any $X \in C_j$. Consider first the case where $\{a_i\} \in C_j$ for some $a_i \in A$. If $X = \{a_i\}$, then $\sum_{x \in X} \omega_X(x).u(x) = u(a_i) = k - j + 1$. If X and $\{a_i\}$ are distinct, then, as $R \in D^{\pi^{KELLY}}(\rho)$, there exist $x, y \in X \setminus \{a_i\}$ such that $x \rho a_i$ and $a_i \rho y$. So $\min_{z \in X} u(z) < u(a_i) < \max_{z \in X} u(z)$ and by Lemma 3.4, there exists $\omega_X \in \Omega_X$ such that $\sum_{x \in X} \omega_X(x).u(x) = u(a_i) = k - j + 1$. Now

consider the case where $\{x\} \in C_j$ for no $x \in A$. Let $i \in \{1, \dots, m\}$ be such that $\{a_d\} P X$ for all $i \in \{1, \dots, i\}$ and $X P \{a_d\}$ for all $d \in \{i+1, \dots, m\}$. As $R \in D^{\pi^{KELLY}}(\rho)$, there exists $x \in X \setminus \{a_i\}$ such that $a_i \rho x$ and there exists $y \in X \setminus \{a_{i+1}\}$ such that $y \rho a_{i+1}$. Thus, $\min_{z \in X} u(z) \leq u(a_{i+1}) = k - f(i+1) + 1$ and $\max_{z \in X} u(z) \geq u(a_i) = k - f(i) + 1$. Moreover, $f(i) < j < f(i+1)$ implying $\min_{z \in X} u(z) < k - j + 1 < \max_{z \in X} u(z)$ which, by Lemma 3.4, implies the existence of $\omega_X \in \Omega_X$ such that $\sum_{x \in X} \omega_X(x).u(x) = k - j + 1$. ■

Remark 3.1 For each $\rho \in \Pi$, we have $\lambda^+(\rho), \lambda^-(\rho) \in D^{\pi^{KELLY}}(\rho)$, hence by Theorem 3.1, $\lambda^+(\rho), \lambda^-(\rho) \in D^\Omega(\rho)$.

The next application of Theorem 3.4 is for BEUC and EEUC, which is a case in point to show that the converse of the inclusion expressed by Theorem 3.4 need not hold.

Theorem 3.6 $D^{\Gamma^{BEUC}}(\rho) \subsetneq D^{\pi^{\Gamma^{BEUC}}}(\rho)$ and $D^{\Gamma^{EEUC}}(\rho) \subsetneq D^{\pi^{\Gamma^{EEUC}}}(\rho) \forall \rho \in \Pi$.

Proof. Take any $\rho \in \Pi$. By Theorem 3.4, we have $D^{\Gamma^{BEUC}}(\rho) \subseteq D^{\pi^{\Gamma^{BEUC}}}(\rho)$ and $D^{\Gamma^{EEUC}}(\rho) \subseteq D^{\pi^{\Gamma^{EEUC}}}(\rho)$. To see that both inclusions are strict, we check that $\lambda^+(\rho) \in D^{\pi^{\Gamma^{BEUC}}}(\rho) \cap D^{\pi^{\Gamma^{EEUC}}}(\rho)$ while $\lambda^+(\rho) \notin D^{\Gamma^{BEUC}}(\rho) \cup D^{\Gamma^{EEUC}}(\rho)$.

$D^{\Gamma^{EEUC}}(\rho)$. As $D^{\pi^{\Gamma^{EEUC}}}(\rho) \subset D^{\pi^{\Gamma^{BEUC}}}(\rho)$ and $D^{\Gamma^{EEUC}}(\rho) \subset D^{\Gamma^{BEUC}}(\rho)$, it suffices to check that $\lambda^+(\rho) \in D^{\pi^{\Gamma^{EEUC}}}(\rho)$ and $\lambda^+(\rho) \notin D^{\Gamma^{BEUC}}(\rho)$. We recall that by Theorem 3.3 $D^{\pi^{\Gamma^{EEUC}}}(\rho) = D^{\pi^{CD}}(\rho)$ and leave checking $\lambda^+(\rho) \in D^{\pi^{CD}}(\rho)$ as an exercise to the reader. To see $\lambda^+(\rho) \notin D^{\Gamma^{BEUC}}(\rho)$, suppose there exists a triple $(\rho, u, \omega) \in \Pi \times U_\rho \times \Omega$ that directly generates $\lambda^+(\rho)$. Take any distinct $a, b, c \in A$ with $a \succ \rho \succ b \succ \rho \succ c$. Note that by definition of the leximax extension, we have $\{a, b, c\} \succ \lambda^+(\rho) \succ \{a, c\} \succ \lambda^+(\rho) \succ \{b\}$. Therefore,

$$\begin{aligned}
 & \frac{1}{\sum_{x \in \{a, b, c\}} \omega_A(x)} \sum_{x \in \{a, b, c\}} \omega_A(x) u(x) > \frac{1}{\sum_{x \in \{a, c\}} \omega_A(x)} \sum_{x \in \{a, c\}} \omega_A(x) u(x) \\
 \Rightarrow & \frac{1}{\sum_{x \in \{a, b, c\}} \omega_A(x)} \left(\omega_A(b) u(b) + \sum_{x \in \{a, c\}} \omega_A(x) u(x) \right) > \frac{1}{\sum_{x \in \{a, c\}} \omega_A(x)} \sum_{x \in \{a, c\}} \omega_A(x) u(x) \\
 \Rightarrow & \frac{\omega_A(b) u(b)}{\sum_{x \in \{a, b, c\}} \omega_A(x)} > \left(\frac{1}{\sum_{x \in \{a, c\}} \omega_A(x)} - \frac{1}{\sum_{x \in \{a, b, c\}} \omega_A(x)} \right) \sum_{x \in \{a, c\}} \omega_A(x) u(x) \\
 \Rightarrow & \frac{\omega_A(b) u(b)}{\sum_{x \in \{a, b, c\}} \omega_A(x)} > \frac{\omega_A(b)}{\sum_{x \in \{a, c\}} \omega_A(x) \sum_{x \in \{a, b, c\}} \omega_A(x)} \sum_{x \in \{a, c\}} \omega_A(x) u(x) \\
 \Rightarrow & u(b) > \frac{1}{\sum_{x \in \{a, c\}} \omega_A(x)} \sum_{x \in \{a, c\}} \omega_A(x) u(x), \text{ contradicting that } Y \succ \lambda^+(\rho) \succ Z, \text{ thus} \\
 & \text{that } (\rho, u, \omega) \text{ directly generates } \lambda^+(\rho). \blacksquare
 \end{aligned}$$

4 A Remark on Strategy-Proof Social Choice Correspondences

The ‘‘strategy-proofness’’ of a social choice correspondence depends on how preferences over alternatives is extended over sets. If this extension is made through expected utility consistency, then the subtleties discussed in the previous section affect the definition of strategy-proofness.

To argue this formally, let $\underline{\rho} = (\rho_1, \dots, \rho_n) \in \Pi^N$ stand for a *preference profile* over A where ρ_i is the preference of $i \in N$. A *social choice correspondence* (SCC) is a mapping $f : \Pi^N \rightarrow \underline{A}$. Consider a set of admissible priors Γ inducing the extension axiom π^Γ . We say that a SCC $f : \Pi^N \rightarrow \underline{A}$ is

- *strategy-proof under* Γ iff given any $i \in N$ and any $\underline{\rho}, \underline{\rho}' \in \Pi^N$ with $\rho_j = \rho'_j \forall j \in N \setminus \{i\}$, we have $f(\underline{\rho}) R f(\underline{\rho}')$ for all $R \in D^\Gamma(\rho_i)$.
- *strongly strategy-proof under* Γ iff given any $i \in N$ and any $\underline{\rho}, \underline{\rho}' \in \Pi^N$ with $\rho_j = \rho'_j \forall j \in N \setminus \{i\}$, we have $f(\underline{\rho}) R f(\underline{\rho}')$ for all $R \in D^{\pi^\Gamma}(\rho_i)$.

At a first glance, the second definition deserves to be qualified as “strong”, because, by Theorem 3.4, we have $D^\Gamma(\rho) \subseteq D^{\pi^\Gamma}(\rho)$ for all $\rho \in \Pi$. Nevertheless, the two definitions coincide, as the following theorem announces:

Theorem 4.1 *Take any non-empty $\Gamma \subseteq \Omega$ inducing the extension axiom π^Γ . A SCC $f : \Pi^N \rightarrow \underline{A}$ strategy-proof under Γ if and only if f is strongly strategy-proof under Γ .*

Proof. Take any non-empty $\Gamma \subseteq \Omega$. The “if” part follows from Theorem 3.4. To show the “only if” part, consider a SCC $f : \Pi^N \rightarrow \underline{A}$ which fails to be strongly strategy-proof. So there exist $i \in N$ and $\underline{\rho}, \underline{\rho}' \in \Pi^N$ with $\rho_j = \rho'_j \forall j \in N \setminus \{i\}$ such that $f(\underline{\rho}') P f(\underline{\rho})$ for some $R \in D^{\pi^\Gamma}(\rho_i)$. Thus $(f(\underline{\rho}), f(\underline{\rho}')) \notin \pi^\Gamma(\rho_i)$, implying the existence of some $\tilde{u} \in U_{\rho_i}$ and some $\tilde{\omega} \in \Gamma$ such that $\sum_{x \in f(\underline{\rho}')} \tilde{\omega}_{f(\underline{\rho}')}(\underline{\rho}') \cdot \tilde{u}(x) > \sum_{x \in f(\underline{\rho})} \tilde{\omega}_{f(\underline{\rho})}(x) \cdot \tilde{u}(x)$. Therefore, letting $\tilde{R} \in$

\mathfrak{R} be directly by $(\rho_i, \tilde{u}, \tilde{\omega})$, there exist $i \in N$ and $\underline{\rho}, \underline{\rho}' \in \Pi^N$ with $\rho_j = \rho'_j \forall j \in N \setminus \{i\}$ such that $f(\underline{\rho}') \tilde{P} f(\underline{\rho})$ for $\tilde{R} \in D^\Gamma(\rho_i)$, showing that f fails to be strategy-proof. ■

Thus, in analyzing the strategy-proofness of SCCs, it does not matter whether orderings over sets are obtained by completing a partial order generated through expected utilities or are directly generated with reference to expected utilities. The literature on strategy-proof SCCs exhibits both definitions of strategy-proofness. For example, Ching and Zhou (2002) use strong strategy-proofness while Barberà, Dutta and Sen (2001) adopt the “weaker” version. We know by Theorem 4.1 that this choice, everything else being equal, does not affect the analysis.¹⁹

On the other hand, it would be no surprise that the choice of the set of admissible priors Γ matters. In fact, it immediately follows from the definitions that expanding Γ can only strengthen strategy-proofness. As a case in point, we have Barberà, Dutta and Sen (2001) who consider strategy-proofness under Γ^{EEUC} and Γ^{BEUC} . They show that under Γ^{EEUC} strategy-proof SCCs are either dictatorial or bidictatorial²⁰ while Γ^{BEUC} admits only dictatorial rules. Hence the fact that $\Gamma^{EEUC} \subset \Gamma^{BEUC}$ matters and strategy-proofness under Γ^{BEUC} is effectively stronger than it is under Γ^{EEUC} . On the other

¹⁹It is worth noting that the analysis of Barberà, Dutta and Sen (2001) is for social choice rules that map preference profiles over sets into sets. These being more general than standard social choice correspondences, their impossibility under Γ^{BEUC} implies the impossibility that Ching and Zhou (2002) establish under Γ^{BEUC} .

²⁰A SCC $f : \Pi^N \rightarrow \underline{A}$ is *dictatorial* iff $\exists i \in N$ such that $f(\underline{\rho}) = \{\arg \max \rho_i\} \forall \underline{\rho} \in \Pi^N$. A SCC $f : \Pi^N \rightarrow \underline{A}$ is *bidictatorial* iff $\exists i, j \in N$ such that $f(\underline{\rho}) = \{\arg \max \rho_i, \arg \max \rho_j\} \forall \underline{\rho} \in \Pi^N$.

hand, Ozyurt and Sanver (2006) pick Γ^{GEUC} as the set of admissible priors and show the equivalence between strategy-proofness and dictatorship. Thus expanding Γ^{EEUC} to Γ^{GEUC} leaves the definition of strategy-proofness intact.

5 Conclusion

We explore the problem of extending a complete order over a set to its power set by the assignment of utilities over alternatives and probability distributions over sets - hence the idea of expected utility consistent extensions. We express three well-known expected utility consistent extensions of the literature as a function of admissible priors and we characterize them in terms of extension axioms which do not refer to the concept of expected utility. Moreover, we display that

- assigning utilities and probabilities which end-up ordering sets according to their expected utilities

and

- completing the partial order determined by the pairs of sets whose ordering is independent of the utility and probability assignment

are different approaches. This difference has an immediate reflection to the analysis of strategy-proof social choice correspondences which we also discuss and clarify. In brief, we present a framework which allows a general and unified exposition of expected utility consistent extensions while it allows to emphasize various subtleties, the effects of which seem to be underestimated - particularly in the literature on strategy-proof social choice correspondences.

6 Appendix A

Lemma 6.1 *For all $\rho \in \Pi$ and all $(X, Y) \in \pi^{GF}(\rho)$ with $X \cap Y \neq \emptyset$ and $X \setminus Y \neq \emptyset$, we have $(X, X \cap Y) \in \pi^{\Gamma^{BEUC}}(\rho)$.*

Proof. Take any $\rho \in \Pi$ and let (X, Y) be as in the statement of the lemma. As $(X, Y) \in \pi^{GF}(\rho)$, we have $x \rho y \forall x \in X \setminus Y \forall y \in Y$, thus $x \rho y \forall x \in X \setminus Y \forall y \in X \cap Y$. Therefore, given any $u \in U_\rho$ and any $\omega \in \Gamma^{BEUC}$, we have $\sum_{x \in X \setminus Y} \omega_{X \setminus Y}(x)u(x) > \sum_{x \in X \cap Y} \omega_{X \cap Y}(x)u(x)$, which implies

$\frac{1}{\sum_{x \in X \setminus Y} \omega_A(x)} \sum_{x \in X \setminus Y} \omega_A(x)u(x) > \frac{1}{\sum_{x \in X \cap Y} \omega_A(x)} \sum_{x \in X \cap Y} \omega_A(x)u(x)$. Multiplying both

sides by $\frac{\sum_{x \in X \setminus Y} \omega_A(x)}{\sum_{x \in X} \omega_A(x)}$ gives

$$\frac{1}{\sum_{x \in X} \omega_A(x)} \sum_{x \in X \setminus Y} \omega_A(x)u(x) > \left(\frac{\sum_{x \in X \setminus Y} \omega_A(x)}{\sum_{x \in X} \omega_A(x)} \right) \frac{1}{\sum_{x \in X \cap Y} \omega_A(x)} \sum_{x \in X \cap Y} \omega_A(x)u(x)$$

$$\Rightarrow \frac{1}{\sum_{x \in X} \omega_A(x)} \sum_{x \in X \setminus Y} \omega_A(x)u(x) > \frac{\sum_{x \in X} \omega_A(x) - \sum_{x \in X \cap Y} \omega_A(x)}{\sum_{x \in X} \omega_A(x) \sum_{x \in X \cap Y} \omega_A(x)} \sum_{x \in X \cap Y} \omega_A(x)u(x)$$

$$\Rightarrow \frac{1}{\sum_{x \in X} \omega_A(x)} \left(\sum_{x \in X \setminus Y} \omega_A(x)u(x) + \sum_{x \in X \cap Y} \omega_A(x)u(x) \right)$$

$$> \frac{1}{\sum_{x \in X \cap Y} \omega_A(x)} \sum_{x \in X \cap Y} \omega_A(x)u(x)$$

$$\Rightarrow \frac{1}{\sum_{x \in X} \omega_A(x)} \sum_{x \in X} \omega_A(x)u(x) > \frac{1}{\sum_{x \in X \cap Y} \omega_A(x)} \sum_{x \in X \cap Y} \omega_A(x)u(x)$$

$$\Rightarrow \sum_{x \in X} \omega_X(x)u(x) > \sum_{x \in X \cap Y} \omega_{X \cap Y}(x)u(x)$$

$$\Rightarrow (X, X \cap Y) \in \pi^{\Gamma^{BEUC}}(\rho). \blacksquare$$

Lemma 6.2 For all $\rho \in \Pi$ and all $(X, Y) \in \pi^{GF}(\rho)$ with $X \cap Y \neq \emptyset$ and $Y \setminus X \neq \emptyset$ we have $(X \cap Y, Y) \in \pi^{\Gamma^{BEUC}}(\rho)$.

Proof. Take any $\rho \in \Pi$ and let (X, Y) be as in the statement of the lemma. As $(X, Y) \in \pi^{GF}(\rho)$, we have $x \rho y \forall x \in X \forall y \in Y \setminus X$, thus $x \rho y \forall x \in X \cap Y \forall y \in Y \setminus X$. Therefore, given any $u \in U_\rho$ and any $\omega \in \Gamma^{BEUC}$, we have $\sum_{x \in X \cap Y} \omega_{X \cap Y}(x)u(x) > \sum_{x \in Y \setminus X} \omega_{Y \setminus X}(x)u(x)$, which implies

$$\frac{1}{\sum_{x \in X \cap Y} \omega_A(x)} \sum_{x \in X \cap Y} \omega_A(x)u(x) > \frac{1}{\sum_{x \in Y \setminus X} \omega_A(x)} \sum_{x \in Y \setminus X} \omega_A(x)u(x)$$

Multiplying both sides by $\frac{\sum_{x \in Y \setminus X} \omega_A(x)}{\sum_{x \in Y} \omega_A(x)}$ gives

$$\frac{\sum_{x \in Y \setminus X} \omega_A(x)}{\sum_{x \in Y} \omega_A(x)} \sum_{x \in X \cap Y} \omega_A(x)u(x) > \frac{1}{\sum_{x \in Y} \omega_A(x)} \sum_{x \in Y \setminus X} \omega_A(x)u(x)$$

$$\begin{aligned}
&\Rightarrow \frac{\sum_{x \in Y} \omega_A(x) - \sum_{x \in X \cap Y} \omega_A(x)}{\sum_{x \in Y} \omega_A(x)} \sum_{x \in X \cap Y} \omega_A(x) u(x) > \frac{1}{\sum_{x \in Y} \omega_A(x)} \sum_{x \in Y \setminus X} \omega_A(x) u(x) \\
&\Rightarrow \frac{1}{\sum_{x \in X \cap Y} \omega_A(x)} \sum_{x \in X \cap Y} \omega_A(x) u(x) \\
&> \frac{1}{\sum_{x \in Y} \omega_A(x)} \left(\sum_{x \in Y \setminus X} \omega_A(x) u(x) + \sum_{x \in X \cap Y} \omega_A(x) u(x) \right) \\
&\Rightarrow \frac{1}{\sum_{x \in X \cap Y} \omega_A(x)} \sum_{x \in X \cap Y} \omega_A(x) u(x) > \frac{1}{\sum_{x \in Y} \omega_A(x)} \sum_{x \in Y} \omega_A(x) u(x) \\
&\Rightarrow (X \cap Y, Y) \in \pi^{\Gamma^{BEUC}}(\rho). \quad \blacksquare
\end{aligned}$$

7 References

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