One-way monotonicity as a form of strategy-proofness

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Abstract Suppose that a vote consists of a linear ranking of alternatives, and that in a certain profile some single ***pivotal voter*** *v* is able to change the outcome of an election from *s* alone to *t* alone, by changing her vote from σ to τ. A voting rule $\mathcal{F}$ is ***two-way monotonic*** if such an effect is only possible when *v* moves *t* from below *s* (according to σ) to above *s* (according to τ). ***One-way monotonicity*** is the strictly weaker requirement that such an effect never occur when *v* makes the opposite switch, by moving *s* from below *t* to above *t*. Two-way monotonicity is a very strong property, equivalent, over any domain, to ***strategy proofness***. It thus cannot be satisfied by any "reasonable" resolute voting rule over the full domain. One-way monotonicity fails for every Condorcet extension; in this respect, and in others, it resembles Moulin's ***participation*** property, although the two properties are independent. One-way monotonicity holds for all ***sensible*** voting rules – those for which the election outcome is determined by the numerical value of a function called a ***sensible virtue***. Total score, as determined by any scoring rule, is a sensible virtue, but the class of sensible rules is larger than that of scoring rules. The names for these monotonicities arise from their interpretations in terms of manipulability. We may think of either σ or τ as representing *v*'s sincere preferences. For a two-way monotonic function, neither of these interpretations ever yields a successful manipulation. For a one-way monotonic rule $\mathcal{F}$, whenever one of the interpretations yields a successful manipulation, the other yields a ***positive response***, in which $\mathcal{F}$ offers *v* a strictly better result when she votes sincerely. For such a rule $\mathcal{F}$, each manipulation can thus be seen as part of the cost to be paid for appropriate responsiveness to the sincere will of the electorate.

Key Words strategy-proofness, manipulation, monotonicity, participation, no-show paradox, scoring rule, sensible virtue, Condorcet extension, resolute voting rule.

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§1 Introduction

A real-valued function *f* is said to be *monotonically increasing* if whenever $x_1 < x_2$ we have $f(x_1) < f(x_2)$ (for *strictly increasing*) or $f(x_1) \leq f(x_2)$ (for *weakly increasing*). Monotonicity properties for voting rules are loosely based on this idea; we say that such a rule $\mathcal{F}$ is monotonic if whenever one or more voters change their votes in a certain "direction," the effect is to move the outcome of the election in a similar direction. A vote, in our context, will consists of a strict preference ranking – a linear ordering without ties – of all alternatives, so it's not completely clear what "direction" means; the variety of possible interpretations leaves room for a number of different monotonicity properties. This is the case even for *resolute* rules, which yield a unique winning alternative for each profile. For *irresolute* rules, in which several alternatives may be tied as winners, the possibilities ramify further. In the current paper we confine ourselves to resolute rules, but take up monotonicity in the irresolute context in the sequel, Sanver and Zwicker [2007].

Of the monotonicity properties considered to date in the voting literature, all are of the weakly increasing, or ≤ type, for a reason that seems at first to be compelling – it is unreasonable to expect the outcome of an election to change each time a few voters change their votes, because the margin of victory by the winner may be too large to be easily overcome. Among these, two major classes stand out.

The first class contains monotonicities that have a normative appeal independent of any strategic concern. *Simple monotonicity,* which has most often simply been called *monotonicity*, has stood virtually alone as a representative of this class. Loosely, it asserts that raising a single alternative *s* in a voter's preferences (while leaving the ranking otherwise unchanged) is never detrimental to *s*'s prospects for winning.[4] Most voting rules considered in the literature satisfy simple monotonicity, but it is known to fail for the following closely related rules: all scoring elimination rules (JH Smith [1973]), including Hare (or "alternative vote"), and plurality run-off. Fishburn [1982] discusses other examples. Simple monotonicity is a rather "weak" monotonicity but it does discriminate, albeit to a limited extent, among reasonable voting rules.

This is in contrast to the second class of monotonicities, wherein the normative appeal rests on strategic considerations. These properties arose as a consequence of explorations of strategy-proofness and implementation. For example, Muller and Satterthwaite [1977] prove that for social choice functions defined over the full domain of preference profiles, strategy-proofness is equivalent to *strong positive association* – a monotonicity condition which Maskin [1977, 1999] showed to be necessary (but not sufficient) for Nash implementability. On the other hand, Nash implementability is equivalent to Danilov [1992] monotonicity which, although generally stronger, is equivalent to Maskin monotonicity for social choice rules that do not admit ties. In this case, a failure of the condition can typically be identified with a situation in which one voter can manipulate

[4] This may be the oldest known monotonicity property. Other names have also been used over its relatively long history (which predates the modern resurrection of social choice theory in Black [1958] – see Brams and Fishburn [2002] and comments on page 120 of Fishburn [1982], including footnote 1).

the outcome so as to obtain a preferred outcome by misrepresenting her preference. On the other hand, we know from Gibbard [1973] and Satterthwaite [1975] that over the full domain of preference profiles, a strategy-proof social choice function whose range contains at least three alternatives is dictatorial. Thus monotonicity properties in this second class are so strong that they hold for no reasonable (resolute) voting rule. While their theoretical importance is significant, they are less useful as a basis for comparing realistic voting systems in terms of manipulability.[5]

Our investigations arise from the following question. Suppose that when the vote of some particular voter v is the ranking σ, alternative s is the sole winner of a certain election, but that when v votes instead for the ranking τ, while all other votes remain unchanged, some different alternative t is the sole winner. Given such a *pivot*, what should "monotonicity" require, in terms of how σ and τ rank the two alternatives s and t, relative to each other?

We introduce here two new monotonicity properties, based on answers to this question. The stronger property, *two-way monotonicity*, requires that the voter v described above must have lifted t from below s in σ, to above s in τ, and falls squarely into the second class, as it is equivalent to strategy-proofness. This property is not of independent interest, but it helps frame the idea for *one-way monotonicity*, its weaker cousin, which requires that v must *not* have lifted s from below t in σ, to above t in τ. One-way monotonicity does appear to be new, and it discriminates usefully among standard voting rules. One-way monotonicity is of "medium" strength, in that it holds of a number of natural voting rules, yet fails of a number of others. We argue that it partakes of some traits from both classes.

The rest of the paper is organized as follows. In §2 we set the context and present necessary background material, while §3 introduces the properties of two-way and one-way monotonicity. As monotonicity and manipulability are often considered to be two sides of the same coin, we examine the interpretation of these new properties in terms of manipulability and strategy-proofness. For one-way monotonicity, in particular, this analysis suggests some novel ways to think about the way a voting rule responds to changes in a voter's expressed preferences.

The *sensible* voting rules we introduce in §4 include all scoring rules and more, and are one-way monotonic. In §5 we turn to electorates of variable size. Brams and Fishburn [1983] introduce the *no-show paradox*, wherein a voter may obtain a preferred outcome by staying home rather than voting. Moulin [1988a, 1988b] shows that the *participation* axiom, which asserts that no no-show paradoxes occur, is satisfied by no Condorcet extension. Despite their difference in contexts (in that participation requires a variable electorate), one-way monotonicity and participation have some important relationships.

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[5] One consequence is that some authors have published *frequency studies*, in which they compare neutral and anonymous voting rules such as the Borda count, or Copeland rule, by constructing one or more numerical measures of how frequent, or probable, are the instances of manipulability or failures of monotonicity. They then apply these measures to versions of the original rule that have been rendered resolute through the use of a fixed ranking (a *tie-breaking agenda*). We give a more detailed discussion of this literature in Section 3.

We establish that participation implies half-way monotonicity, a weak form of one-way monotonicity, and that the converse holds for voting rules satisfying *homogeneity* and *reversal cancellation*. As stand-alone properties, however, one-way monotonicity and participation are independent.

In §6 we present our main negative result. By exploiting the parallels between participation and one-way monotonicity, and introducing some of the ideas from §5, we are able to elaborate on Moulin's argument and show that no Condorcet extension is one-way monotonic. Hare's rule also fails one-way monotonicity, as does the closely related plurality run-off rule, albeit with a small qualification. These results are the basis for our claim that one-way monotonicity discriminates usefully among plausible voting rules.

In the concluding section 7 we point to future areas of research, including that of further clarifying the relationship among the various monotonicity properties in the context of irresolute voting rules that are both neutral and anonymous. This issue bears directly on the methodology we use throughout the paper, of rendering all voting rules resolute by employing a fixed tie-breaking agenda.

§2 Basic notions

Let $N = \{i, j, \ldots\}$ be a finite set of n ***voters*** and $A = \{s, t, \ldots\}$ be a finite set of $m \geq 3$ ***alternatives***. A ***profile*** R for N consists of an assignment, to each $i \in N$, of a ***linear ranking*** (strict linear ordering) $\sigma = R(i)$ of A. We'll write $s <_\sigma t$ to indicate that a voter with ranking σ strictly prefers t to s, and depict such a situation by placing t higher than s in σ:

$$\begin{array}{c} \underline{\sigma} \\ \vdots \\ t \\ \vdots \\ s \\ \vdots \end{array}$$

We'll use $\leq_\sigma$ to denote the corresponding weak linear order on A.

A ***social choice rule*** or ***voting rule*** is a function $\mathcal{F}$ that returns, for each profile R for N, a nonempty set $\mathcal{F}(R) \subseteq A$. We think of the alternatives in $\mathcal{F}(R)$ as being the winners of the election represented by R, so that $|\mathcal{F}(R)| > 1$ would indicate a tie among the several alternatives in $\mathcal{F}(R)$. A ***resolute*** voting rule $\mathcal{F}$ is one for which there are never ties: $|\mathcal{F}(R)| = 1$ for every profile R, and we write $\mathcal{F}(R) = s$ in place of $\mathcal{F}(R) = \{s\}$. An ***irresolute*** rule $\mathcal{F}$ is one that is not *required* to be resolute. The domain for a voting rule $\mathcal{F}$ will most often consist of the ***full domain*** of all possible profiles for some fixed (but unspecified) N. However, at various times we will impose conditions on the number of voters, or will assume that a single rule applies to a *variable electorate* (discussed in §5), or will refer to a restricted domain in which only certain profiles for *N* are allowed. When we say, then, that one property *P* of voting rules *implies* a second property *Q*, we may mean that every type-*P* rule on the full domain is a type-*Q* rule. A stronger assertion would be that for *every* domain $\mathcal{D}$, every type-*P* rule on $\mathcal{D}$ is a type-*Q* rule, and an assertion of intermediate strength might state that the implication holds for every domain that is "sufficiently rich."

For many voting applications, the properties of ***neutrality*** (all alternatives are treated equally) and ***anonymity*** (all alternatives are treated equally) would be considered to be absolute minimum fairness requirements.[6] However, except in certain special circumstances (see Moulin [1988a], exercise 9.9 on 252-253) voting rules that are both anonymous and neutral cannot also be resolute, and this creates some difficulty, because defining monotonicity or manipulability for irresolute rules is notoriously difficult.

We will sidestep this problem in one of the standard ways, by breaking all ties for any given irresolute rule. If $\prec$ is any ***fixed agenda*** (that is, a strict ranking of the alternatives) and $\mathcal{F}$ is a voting rule, $\mathcal{F}^{\prec}$ will denote the resolute rule obtained from $\mathcal{F}$ by setting

$$\mathcal{F}^{\prec}(R) = \prec\text{-}max[\mathcal{F}(R)]$$

where $\prec$*-max[S]* is the $\prec$-maximal element of S for any nonempty $S \subseteq A$. We can think of $\prec$ as representing the preference ranking of a ***tie-breaking dictator*** who is ***absent***, in that she is not one of the voters.

Of course, imposing such a tie-breaking agenda on an anonymous and neutral rule $\mathcal{F}$ typically destroys neutrality (while preserving anonymity). Can this approach, then, ever yield useful information about monotonicity properties for fair voting rules? Surprisingly, the answer is *yes* – but a satisfactory explanation requires a rather detailed examination of monotonicity properties for irresolute rules and their relationships to tie-breaking agendas. We say a bit more about this in the concluding section, and refer the reader to Sanver and Zwicker [2007] for details.[7]

A number of voting rules make use only of the pairwise majority information in a profile. Given any profile *R*, and alternatives *s* and *x*, let $\#_R(s > x)$ denote the number of voters who rank *s* over *x*, and $Net_R(s > x)$ denote $\#_R(s > x) - \#_R(x > s)$, the ***net pairwise majority*** of *s* over *x*. Of course, $Net_R(s > x) > 0$ holds exactly when a strict majority of voters ranks *s* over *x*. If *R* is a profile and *s* is an alternative satisfying that for each alternative $x \neq s$, a strict majority of voters rank *s* over *x*, then we say that *s* is a ***Condorcet winner***. We'll use $\mathcal{D}_{Condorcet}$ to denote the Condorcet domain, consisting of those profiles for

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[6] For precise definitions of neutrality and anonymity, see (for example) Moulin [1988a].

[7] There is a second reason to be suspicious of the tie-breaking agenda approach. When we observe in §4 and §6, that (after ties are broken by an agenda) the Borda count is one-way monotonic and the Copeland rule is not, are we learning anything about the innate properties of these two voting rules? Might it be the case that these results tell us more about the smoothness with which these rules mate with tie-breaking agendas, than they do about one-way monotonicity? We refer the reader to the conclusion, and to Sanver and Zwicker [2007] for further discussion.

which a Condorcet winner exists. Condorcet winners are, of course, unique on $\mathcal{D}_{Condorcet}$, and the voting rule that picks the Condorcet winner over this domain is known as the ***Condorcet rule***, or ***pairwise majority rule***. Any voting rule that is defined over the full domain and agrees with the Condorcet rule on $\mathcal{D}_{Condorcet}$ is a ***Condorcet extension*** (or is ***Condorcet consistent***).

The ***Copeland score*** of an alternative s is simply the number of alternatives x satisfying $Net_P(s > x) > 0$, and the ***Copeland rule*** is the Condorcet extension that chooses the set of alternatives with maximal Copeland score. The ***net Simpson score*** of an alternative s for profile R is

$$\boldsymbol{t_R^*(s)} = Min\{Net_R(s > x) \,/\, x \neq s\},$$

We'll freely drop any "R" subscript when context allows. The ***Simpson rule*** chooses the set of alternatives with maximal Simpson score; it is another Condorcet extension, as $t^*(s) > 0$ is satisfied if and only if s is a Condorcet winner. Both Copeland and Simpson are ***net pairwise rules***, in that they make use only of the information contained in the net pairwise majorities.

Scoring rules constitute a second important class of voting rules. A vector $\langle w \rangle = \langle w_1, w_2, \ldots, w_m \rangle$ is a ***vector of scoring weights*** provided that the w_i are real numbers satisfying $w_1 \geq \ldots \geq w_m$. Such a vector is ***proper*** if $w_1 > w_m$ and is ***strict*** if $w_1 > w_2 > \ldots > w_m$. Every vector $\langle w \rangle$ of scoring weights induces a corresponding ***scoring rule***, as follows: each voter assigns w_1 points to her top-ranked alternative, w_2 to her second-ranked, etc., and the rule chooses the set of alternatives with maximal score (where the ***score of an alternative*** s is the sum of all points awarded to s by all voters). We'll say that the scoring rule is ***strict*** (or ***proper***) according to whether it is induced by some strict (or proper) vector. The best-known scoring rules include the ***plurality rule***, which is induced by the vector $\langle 1, 0 \ldots, 0 \rangle$, so that the ***plurality score*** of an alternative s is simply the number of voters who top-rank s; the ***anti-plurality rule***, which is induced by the vector $\langle 0, 0 \ldots, 0, -1 \rangle$, so that the anti-plurality score of an alternative s is minus the number of voters who bottom-rank s; and the ***Borda count***, which is induced by the vector $\langle m-1, m-3, \ldots, 1-m \rangle$[8], so that the ***Borda score*** of an alternative s is equal to the sum

$$\sum_{x \in A - \{s\}} Net(s > x)$$

of the pairwise net majorities for s (see, for example, Zwicker [1991]). The Borda count is thus a net pairwise rule (but is well known to not be a Condorcet extension).

[8] For the Borda count, it would suffice to use any other vector of scoring weights with a fixed positive difference $k = w_i - w_{i+1} > 0$ between every pair of adjacent scoring weights.

§3 Monotonicity properties and strategy-proofness

Let $v \in N$, P be a profile for the set N - $\{v\}$, and σ be any strict ranking of A. Then $\boldsymbol{P \wedge \sigma}$ denotes the profile for N obtained from P by adding v's vote for σ; i.e., for each $i \in N$

$$(P \wedge \sigma)(i) = \begin{cases} P(i), & if \;\; i \neq v \\ \sigma, & if \;\; i = v \end{cases}$$

Definition 3.1 A ***focus*** for a resolute social choice function $\mathcal{F}$ is a vector $(P, v, \sigma \rightarrow s, \tau \rightarrow t)$ satisfying:

- $v \in N$ is a voter and P is a profile for the set N - $\{v\}$,
- σ and τ are strict rankings of A, and s and t are alternatives in A,
- $\mathcal{F}(P \wedge \sigma) = s$, *and*
- $\mathcal{F}(P \wedge \tau) = t$.

If we additionally require that $s \neq t$, $(P, v, \sigma \rightarrow s, \tau \rightarrow t)$ is a ***pivot.***

Thus, a pivot is a situation wherein a ***pivotal voter*** v has a choice between voting σ and voting τ, and this choice affects the outcome of the election.

Definition 3.2 A resolute social choice function $\mathcal{F}$ is ***two-way monotonic*** if every pivot $(P, v, \sigma \rightarrow s, \tau \rightarrow t)$ for $\mathcal{F}$ satisfies $s >_\sigma t$ and $t >_\tau s$.

Two-way monotonicity thus asserts that whenever the effect of a pivotal voter's change in vote from σ to τ is to switch the winner of the election from s to t, the pivotal voter must have raised t from below s according to σ, to above s according to τ (see Figure 1). A seemingly weaker requirement, then, would be to insist that whenever the effect of a pivotal voter's change in vote from σ to τ is to switch the winner of the election from s to t, the pivotal voter must <u>**not**</u> have dropped t from above s according to σ, to below s according to τ (see Figure 2). Equivalently, we may simply replace the "and" (from definition 3.2) with an "or":

Definition 3.3 A resolute social choice function $\mathcal{F}$ is ***one-way monotonic*** if every pivot $(P, v, \sigma \rightarrow s, \tau \rightarrow t)$ for $\mathcal{F}$ satisfies $s >_\sigma t$ or $t >_\tau s$.

Note that as a voter moves from σ to τ, neither of these two properties restricts any change in position of alternatives other than s or t (relative to each other, or to s and t). Thus two-way monotonicity requires that any change in election outcome between s and t depend solely on a voter's switch in the relative ranking of s and t, while one-way monotonicity asks, more moderately, that a change in election outcome between s and t not be in total opposition to such a switch. One might say that the smell of Arrovian *I.I.A.* (*independence of irrelevant alternatives*) is stronger for two-way monotonicity.

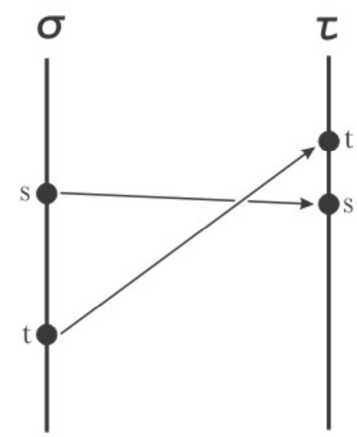

Figure 1 ***Two-way monotonicity***
If a change in vote from σ to τ switches the winner of the election from s to t, then the pivotal voter must have raised t from below s according to σ, to above s according to τ.

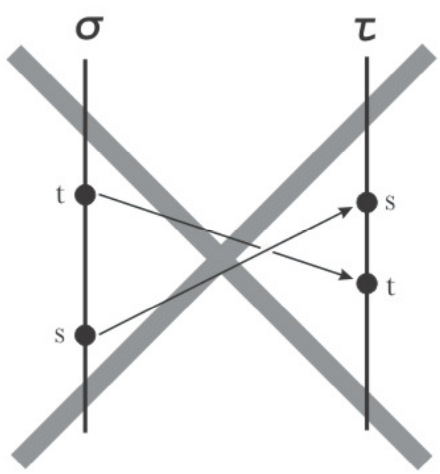

Figure 2 ***One-way monotonicity***
If a change in vote from σ to τ switches the winner of the election from s to t, then the pivotal voter must **not** have raised s from below t according to σ, to above t according to τ.

A resolute rule is *strategy-proof* if no voter can obtain a strictly preferred outcome by changing her current vote (which we interpret as expressing her sincere preferences) to a different ranking (which we interpret as expressing preferences that are not sincere). It is easy to see, then, that in the language of pivots, we can express the concept as follows:

Definition 3.4 A resolute social choice function $\mathcal{F}$ is ***strategy-proof*** if every pivot $(P, v, \sigma \rightarrow s, \tau \rightarrow t)$ for $\mathcal{F}$ satisfies $s >_\sigma t$.

It should be clear from 3.2 that two-way monotonicity is equivalent to strategy-proofness. Furthermore, the equivalence is of the "strong" domain-independent kind discussed earlier. Thus "two-way monotonicity" could be substituted for "strategy-proofness" in any of the remarks that follow.

If two-way monotonicity amounts to full strategy-proofness, what are the implications of one-way monotonicity? We have defined a pivot $(P, v, \sigma \rightarrow s, \tau \rightarrow t)$ for a resolute voting rule $\mathcal{F}$ to be an *ordered* tuple, and we will occasionally speak as if σ were the *initial* choice of voter *v*, who at some later time switches her choice from σ to τ. This implied order of *first* σ*, then* τ is purely a matter of notational or interpretational convenience, however, and plays no role in the definitions of one-way or two-way monotonicity (which remain the same if we switch to the oppositely ordered pivot $(P, v, \tau \rightarrow t, \sigma \rightarrow s)$ while leaving the rest of the definition alone). In other words, such a pivot might as well be considered to be a type of unordered fragment of $\mathcal{F}$'s graph as a function, containing just two inputs with their corresponding outputs.

We may, if we wish, impose an interpretation on such a fragment, by supposing that one of σ or τ represents the sincere preferences of the pivotal voter *v*, and the other represents an attempt at manipulation, but there is nothing inherent in the pivot or the voting rule that points to one of these interpretations over the opposite one. Notice that if we interpret σ as the sincere preferences and τ as the attempted manipulation, then the attempt to manipulate is successful if and only if $s <_\sigma t$; the opposite interpretation represents a successful attempt at manipulation if and only if $t <_\tau s$.

Now consider the following assertion:

"For no pivot for $\mathcal{F}$ does either interpretation ever represent a successful attempt at manipulation."

The following equivalent version

"Every pivot for $\mathcal{F}$ satisfies both $t <_\sigma s$ and $s <_\tau t$."

is simply the definition of two-way monotonicity for $\mathcal{F}$.

On the other hand, the statement:

"For no pivot for $\mathcal{F}$ do the two interpretations ever <u>both</u> represent successful attempts at manipulation."

is equivalent to

"Whenever one of the two interpretations of some pivot for $\mathcal{F}$ represents a successful attempt at manipulation, the opposite interpretation represents an <u>unsuccessful</u> attempt."

and is also equivalent to

"Every pivot for F satisfies either $t <_\sigma s$ or $s <_\tau t$."

which is the definition of one-way monotonicity for $\mathcal{F}$.

But in what we are calling an "unsuccessful attempt at manipulation," the pivotal voter v does strictly worse by voting insincerely than by indicating her true preferences. Arguably, such a situation represents a disincentive to manipulate the rule $\mathcal{F}$. Alternately, it represents a positive incentive to vote sincerely (rather than merely the absence of a disincentive), and so we'll refer to such a situation as a ***strictly positive response*** for $\mathcal{F}$.

This line of reasoning leads us to the following interpretation of one-way monotonicity for a voting rule $\mathcal{F}$:

Every example of a manipulation of $\mathcal{F}$ is also an example of a strictly positive response when interpreted in the "opposite order."

In this sense, for a one-way monotonic rule any instance of manipulability can be seen as part of the cost of doing business – a payment made in order to respond appropriately to the will of the electorate.

Next, we compare our new monotonicities to ones already in the literature.

Definition 3.5 A resolute voting rule $\mathcal{F}$ is ***Maskin monotonic***[9] if for every pivot $(P, v, \sigma \to s, \tau \to t)$, τ differs from σ by moving some alternative from below s in σ to above s in τ.

As Muller and Satterthwaite (1977) show, Maskin monotonicity is equivalent to strategy-proofness over the full domain. However, over arbitrary domains, strategy-proofness is generally stronger.[10] It is also well-known that the Gibbard (1973) and Satterthwaite (1975) result about the equivalence between strategy-proofness and dictatoriality need not hold over arbitrary domains. As noted earlier, "strategy-proofness" can everywhere be replaced by "two-way monotonicity".

One of the best-known monotonicity properties is often just called "monotonicity":

Definition 3.6 A resolute voting rule $\mathcal{F}$ is ***simply monotonic*** if for every profile P with winning alternative s, if one voter changes by moving s up in her ranking (while making no changes in the relative order of the other alternatives) then s remains the winner.

Simple monotonicity has no whiff of I.I.A. Here is an equivalent formulation: For every pivot $(P, v, \sigma \to s, \tau \to t)$, τ differs from σ either by moving some alternative from below s in σ to above s in τ, or by changing the relative order of some two alternatives other than s.

We note that simple and Maskin monotonicity share two features. First, the alternative that is moved "from below s in σ to above s in τ" need not be the new winner t. Second,

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[9] This apparently weak definition of Maskin monotonicity implies the standard version on a broad variety of domains. For details, see footnotes 11 and 22.

[10] One can see Barberà [2001] for the result over arbitrary domains.

allowing several voters to change their votes simultaneously does not strengthen the property.[11] In these respects, these two monotonicities resemble each other more closely than they do two-way or one-way monotonicities. Of course, Maskin monotonicity (hence two-way monotonicity) implies simple monotonicity over any domain. On the other hand, one-way monotonicity and simple monotonicity are logically independent as we formally state and show below:

Proposition 3.7 One-way monotonicity and simple monotonicity are logically independent.

Proof: The Copeland rule satisfies simple monotonicity, as does any net pairwise rule, provided that the rule's reliance on net pairwise majorities is itself monotonic in the appropriate sense. We show in Section 6, however, that Copeland fails to be one-way monotonic. In Section 4, Example 4.11 exemplifies a social choice rule which is one-way monotonic but not simply monotonic. ∎

Properties such as Maskin monotonicity essentially hold of no reasonable voting rules, while simple monotonicity holds of virtually every such rule. One consequence is that rather than basing comparisons on which monotonicity properties are or are not *absolutely* satisfied by various voting rules, some authors (see for example Aleskerov and Kurbanov [1999], D Smith [1999], Favardin et al [2002], and Favardin and Lepelley[2004][12]) have considered *relative* strategy-proofness measures, based on how common are instances of manipulability, or on the probability that a randomly chosen profile is manipulable. We'll refer to such comparisons as *frequency studies*. In effect, such studies count how often a given rule $\mathcal{F}$ "reacts badly" to a change in input, or they measure the probability of a bad reaction.[13]

The papers on frequency studies share the goal of comparing a variety of common voting rules in terms of their degree of manipulability, and a number employ the following methodology: first, some "reasonable" voting rule (in particular, one that is neutral and anonymous) is rendered resolute through the use of a fixed agenda to break ties. Next, some measure[14] is taken of what fraction of profiles are manipulable, for the resolute rule.

Suppose, for the sake of argument, we were to apply this method to the (non-resolute) omninominator rule which, at each profile, picks all alternatives ranked as the best by at

[11] We introduce the "coalitional" forms of some monotonicity properties in 4.9 and 6.1 (see also 6.2 and its footnote). In terms of that language, what we are asserting here is that the strong coalitional form of Maskin monotonicity is no stronger than the definition 3.5 version we define here, and that a similar statement holds for simple monotonicity.

[12] A more complete list appears in the bibliography of Pritchard and Wilson [2006].

[13] It is worth mentioning that Sen (1995) proposes a quite different way of evaluating the extent of non-monotonicity of social choice functions, by extending them minimally to social choice correspondences that are Maskin monotonic.

[14] The measures used vary considerably, both with respect to the underlying probability distribution over profiles, and with respect to other matters, such as whether or not to take account of the number of voters who can manipulate at a given profile.

least one voter.[15] Further suppose that we limit ourselves to the three alternatives a, b, and c; require the number of voters to be more than a handful; and use the tie-breaking agenda $a \succ b \succ c$. Now consider any profile in which no voter (except possibly for our focal voter v) ranks alternative a at the top of his or her ballot and at least one voter (other than v) has b at the top. If v's sincere preferences are $c \succ a \succ b$ then a sincere vote leaves b as the winning alternative, while an insincere vote for $a \succ c \succ b$ (or $a \succ b \succ c$) will elect alternative a, improving the result for v. However, it is easy to see that (under the conditions specified) no other scenario leads to a manipulable profile.

Now under any reasonable choice of probability distribution over profiles, the probability that *no* voter other than v ranks a at the top decreases very rapidly as a function of the number n of voters – so rapidly, in fact, that the probability that a randomly drawn profile is manipulable decreases more quickly for the omninominator rule[16] than for any of the more standard rules considered in the existing frequency studies. It is clear, however, that omninominator scores so well in terms of manipulability for a very simple reason. It is almost never manipulable, because it almost never reacts at all to a change in a single vote; almost every profile leads to a three-way tie among a, b, and c, broken in favor of a by our agenda. We can think of the root cause as being the lack of *responsiveness* of the omninominator rule, but lack of *decisiveness*, in the form of many three-way ties, also plays an important role.

What, after all, is the job of a voting rule? It is to respond to the will of the electorate by making decisions about the winners of elections. The well-known impossibility results of social choice theory tell us how difficult it is to make these decisions without being caught up in some form of inconsistency. So it should not surprise us that a voting rule such as omninominator can avoid making a large number of "bad" decisions by avoiding the making of many decisions at all. Of course, the omninominator rule represents an extreme case. Do such considerations play any role in comparisons among the voting systems actually considered in the published frequency studies? This is a matter for future study, but we point out that some of the rules considered in these studies have many more ties than others (although none is as indecisive as omninominator).

To return to the metaphor with which we began the introduction, these studies measure how often a rule fails to be monotonic in the ≤ sense, but not how often it succeeds in being monotonic in the < sense. Our study of one-way monotonicity raises the possibility that some rules may pay a cost for looking better in the ≤ sense – part of that cost may be that the rule to scores less well in the < sense (which corresponds to what we have called a positive response). So we may learn something interesting about the comparative strengths and weaknesses of various voting rules by measuring the

[15]The omninominator rule was not studied by these authors, presumably because it has little appeal for many of the more common voting applications.

[16] For example, under the *Independent Culture*, or "IC," assumption (see Berg [1985]), with the assumptions described above, the probability that a profile is manipulable under the omninominator rule decreases exponentially, as $\left(\frac{2}{3}\right)^{n-1}$.

frequency of pre-agenda ties, and of post-agenda positive responses, in parallel with that of manipulable profiles.

§4 Positive results: virtues and sensible rules

We consider voting rules that can be characterized in terms of certain type of "score" that generalizes the score from a scoring rule. All such rules are one-way monotonic.

Definition 4.1 Given a finite set N of voters, and a set A of three or more alternatives, a ***virtue*** is a function $\mathcal{V}$ that returns a real number $\mathcal{V}_R(x)$, for each combination of a profile R and an alternative x. Any virtue $\mathcal{V}$ yields an ***induced voting rule*** $\mathcal{F}_{\mathcal{V}}$, which declares the social choice for any profile R to be the set of alternatives x that maximize $\mathcal{V}_R(x)$. This rule $\mathcal{F}_{\mathcal{V}}$ need not be resolute.

Example 4.2 Examples of virtues include:

(i) For each scoring rule with associated vector ⟨w⟩ of scoring weights, let $\mathcal{V}^{\langle w\rangle}{}_R(x)$ denote the total score achieved by an alternative, using the given weights. The induced rule is, of course, the scoring rule for this vector.

(ii) $\mathcal{V}^{OMNINOMINATOR}{}_R(x)$ = 1, if at least one voter top-ranks alternative x
= 0, otherwise.

The induced rule is the omninominator rule (see, for example, Taylor [2005]), wherein the winners are all alternatives that are "nominated" by being top-ranked by at least one voter.

(iii) $\mathcal{V}^{OMNIVETOER}{}_R(x)$ = -1, if at least one voter bottom-ranks alternative x
= 0, otherwise.

The induced rule is the omnivetoer rule, wherein the winners are all alternatives that are not "vetoed" by being ranked at the very bottom by at least one voter (or are all alternatives, if each alternative is bottom-ranked by at least one voter).

(iv) Let $\mathcal{V}^{COPELAND}{}_R(x)$ be the ***Copeland score***. Then the ***Copeland rule*** is the induced system.

In the absence of any further restrictions, the virtue concept is clearly an empty shell; *any* voting rule $\mathcal{F}$ is induced by the following trivial virtue:

$\mathcal{V}^{\mathcal{F}\,TRIVIAL}{}_R(x)$ = 1, if x is chosen by $\mathcal{F}$ at R
= 0, otherwise.

So we see from (i) that a voting rule (e.g., the Borda count) can be induced by two quite different virtues.

Definition 4.3 Suppose $\mathcal{V}$ is a virtue. Given any focus $\Xi = (P, v, \sigma \to s, \tau \to t)$, and any alternative x, let $\boldsymbol{\Delta x}_{\mathcal{V}}$ (or just $\boldsymbol{\Delta x}$) denote $\mathcal{V}_{P \wedge \tau}(x) - \mathcal{V}_{P \wedge \sigma}(x)$, x's change in virtue. If x and y are distinct alternatives, we'll say that ***Ξ lifts x over y*** if $x <_{\sigma} y$ *and* $y <_{\tau} x$. Then $\mathcal{V}$ is ***strictly sensible at Ξ*** if for every pair x, y of distinct alternatives such that Ξ lifts x over y, $\Delta x_{\mathcal{V}} > \Delta y_{\mathcal{V}}$, and $\mathcal{V}$ is ***sensible at Ξ*** if for every pair x, y of distinct alternatives such that Ξ lifts x over y, $\Delta x_{\mathcal{V}} \geq \Delta y_{\mathcal{V}}$. Finally, $\mathcal{V}$ is ***strictly sensible*** if it is strictly sensible at every focus, and is ***sensible*** if it is sensible at every focus.

Notice that sensibility potentially makes demands of all alternatives at a focus (not just of the focal alternatives s and t). The following results are straightforward; proofs are left to the reader.

Proposition 4.4 The following virtues are sensible:

(i) $\mathcal{V}^{\langle w \rangle}$, for every vector $\langle w \rangle$ of scoring weights.

(ii) Any positive linear combination $\lambda_1 \mathcal{V}_1 + \ldots + \lambda_k \mathcal{V}_k$ of sensible virtues $\mathcal{V}_1, \ldots, \mathcal{V}_k$. (The real scalars λ_i are required to be positive.)

(iii) Any sum $\mathcal{V} + \mathcal{W}$ of a sensible virtue $\mathcal{V}$ with an *initial endowment* $\mathcal{W}$. (Every assignment $c = \langle c_y \rangle_{y \in A}$ of real numbers to alternatives gives rise to an ***initial endowment*** virtue $\mathcal{W}$ for which $\mathcal{W}_R(x) = c_x$ for every profile R.)

With respect to Example 4.2, it is straightforward to check that

- $\mathcal{V}^{\langle w \rangle}$ is strictly sensible for every strict vector $\langle w \rangle$ of scoring weights.
- $\mathcal{V}^{OMNINOMINATOR}$, and $\mathcal{V}^{OMNIVETOER}$ are sensible.
- $\mathcal{V}^{COPELAND}$ and $\mathcal{V}^{BORDA\text{-}TRIVIAL}$ are not sensible.

Definition 4.5 A voting rule $\mathcal{F}$ is ***sensible*** (respectively, ***strictly sensible***) if there exists a sensible (respectively, strictly sensible) virtue that induces it (in the sense of 4.1).

Note that the Borda count is induced by a sensible virtue and also by a non-sensible virtue. However, the existence of the former suffices to qualify the Borda count as a sensible voting rule.

Theorem 4.6 Every resolute and sensible voting rule $\mathcal{F}$ is one-way monotonic.

Proof Suppose $\mathcal{F} = \mathcal{F}_{\mathcal{V}}$ where $\mathcal{V}$ is sensible. Let $(P, v, \sigma \to s, \tau \to t)$ be a pivot for $\mathcal{F}$, and assume by way of contradiction that $t >_{\sigma} s$ and $s >_{\tau} t$. As $\mathcal{V}$ is sensible, $\Delta s \geq \Delta t$. As $\mathcal{F}(P \wedge \sigma) = s$, $\mathcal{V}_{P \wedge \sigma}(s) > \mathcal{V}_{P \wedge \sigma}(t)$. Then $\mathcal{V}_{P \wedge \tau}(s) = \mathcal{V}_{P \wedge \sigma}(s) + \Delta s > \mathcal{V}_{P \wedge \sigma}(t) + \Delta t = \mathcal{V}_{P \wedge \tau}(t)$, which contradicts $\mathcal{F}(P \wedge \tau) = t$. ∎

Not every one-way monotonic voting rule is sensible, as we show at the end of the section. Theorem 4.6 leads to the following corollary:

Corollary 4.7 For every sensible voting rule $\mathcal{F}$ and fixed agenda $\prec$, $\mathcal{F}^{\prec}$ is one-way monotonic.

Proof If $\mathcal{F} = \mathcal{F}_{\mathcal{V}}$ where $\mathcal{V}$ is sensible, then $\mathcal{F}^{\prec} = \mathcal{F}_{\mathcal{V}+\mathcal{W}}$, where is $\mathcal{W}$ is the initial endowment given by any assignment $\langle c_y \rangle_{y \in A}$ of real numbers chosen so as to induce the $\prec$ order and so as to be too small to reverse any strict order relation induced by $\mathcal{V}$. More precisely, we require

- $x \prec y$ iff $c_x < c_y$, for each $x, y \in A$, and
- $\mathcal{V}_R(x) < \mathcal{V}_R(y) \Rightarrow \mathcal{V}_R(x) + c_x < \mathcal{V}_R(y) + c_y$, for each $x, y \in A$ and each profile R for N.

By 4.4, $\mathcal{V} + \mathcal{W}$ is sensible, and so $\mathcal{F}^{\prec}$ is one-way monotonic. ∎

Consequences of these results include:

- every scoring rule$^{\prec}$ is one-way monotonic, and
- the omninominator$^{\prec}$ and omnivetoer$^{\prec}$ rules are one-way monotonic.

The omninominator and omnivetoer virtues (and rules) may seem at first to be isolated special cases of sensible virtues that don't correspond to scoring rules. Are they pieces of a larger picture? If $\mathcal{V}^{\langle w \rangle}{}_R(x)$ represents a scoring virtue, and $K \geq 0$ is a constant, then a ***truncated scoring virtue*** will be any virtue of the form $\mathcal{V}_R(x) = Min(K, \mathcal{V}^{\langle w \rangle}{}_R(x))$. Clearly $\mathcal{V}^{OMNIVETOER}$ and $\mathcal{V}^{OMNIVETOER}$ are truncated versions of plurality score and anti-plurality score, respectively. Truncated scoring virtues induce scoring rules with a threshold.[17] Yet with *4* or more alternatives, there are truncated scoring virtues (such as truncated Borda count) whose induced rules are not one-way monotonic.

Some additional insight can be gained through the following stronger version of sensibility. A virtue $\mathcal{V}$ is ***absolutely sensible at a focus*** Ξ if for every pair x, y of distinct alternatives such that Ξ lifts x over y, $\Delta x_{\mathcal{V}} \geq 0$ and $\Delta y_{\mathcal{V}} \leq 0$; $\mathcal{V}$ is ***absolutely sensible*** if it is absolutely sensible at every focus, and a voting rule $\mathcal{F}$ is ***absolutely sensible*** if there exists at least one absolutely sensible virtue that induces $\mathcal{F}$. Clearly, any absolutely sensible virtue or rule is sensible. Now it is straightforward to prove the following analogue to proposition 4.4:

Proposition 4.8 The following virtues are absolutely sensible:

(i) $\mathcal{V}^{PLURALITY}$ and $\mathcal{V}^{ANTI\text{-}P}$

(ii) Any positive linear combination of absolutely sensible virtues.

(iii) Any sum $\mathcal{V} + \mathcal{W}$ of an absolutely sensible virtue $\mathcal{V}$ with an initial endowment $\mathcal{W}$.

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[17] A detailed treatment of scoring rules with a threshold can be found in Saari [1990]. Moreover, Erdem and Sanver [2005] show that minimal Maskin monotonic extensions of scoring rules can be expressed in terms of scoring rules with a threshold that varies as a function of the preference profile.

(iv) Any *original* and *monotonic* transform $f \circ \mathcal{V}$ of an absolutely sensible virtue $\mathcal{V}$ (where $f: \mathbf{R} \to \mathbf{R}$ is ***original*** if $f(0) = 0$ and is ***monotonic*** if $a \geq b \Rightarrow f(a) \geq f(b)$, for all $a, b \in \mathbf{R}$).

Note that truncation is a special case of an original monotonic transform. Thanks to 4.8(iv) (which has no analogue in proposition 4.4) we may add, to our list of one-way monotonic voting rules, examples such as the following:

- the rule induced (after imposition of a tie-breaking agenda) by the virtue $10[\mathcal{V}^{PLURALITY}]^3 + \mathcal{V}^{\langle w \rangle} + 11\,\mathcal{V}^{OMNIVETOER}$, for any vector $\langle w \rangle$ of scoring weights.

For four or more alternatives, if $\langle w \rangle$ is the Borda count vector then $\mathcal{V}^{\langle w \rangle}$ is sensible but not absolutely sensible. However, with three alternatives *every* scoring vector is equivalent to[18] some positive linear combination of the plurality and antiplurality vectors, so that every $\mathcal{V}^{\langle w \rangle}$ is absolutely sensible, by 4.8(ii).

The following example appears in Campbell and Kelly [2002]: Given a profile *P*, take each possible ranking and divide the number of voters who chose that ranking by *2*, dropping any fractional part, to obtain an ***induced*** profile $^1/_2P$ Apply plurality rule to $^1/_2P$, and then break ties using any fixed agenda. If we set aside the tie-breaking step, their rule is induced by the following virtue: $\mathcal{C\text{-}K}_P(x) = \mathcal{V}^{PLURALITY}{}_{(1/2)P}(x)$. It is easy to see that $\mathcal{C\text{-}K}$ is absolutely sensible, and it follows from Corollary 4.7 that their rule is one-way monotonic. Interestingly, $\mathcal{C\text{-}K}$ is not a virtue that is "generated" by the closure properties of propositions 4.4 and 4.8.

We close the section by showing that not every one-way monotonic rule is sensible. We first establish that sensibility implies a coalitional version of one-way monotonicity, defined as follows:

Definition 4.9 A resolute voting rule is ***weakly coalitional one-way monotonic*** if whenever a group of voters having a common preference ranking σ simultaneously all switch their rankings to some common ranking τ, and the effect is to switch the winning alternative from *s* alone to *t* alone, then either $t <_\sigma s$ or $s <_\tau t$. Such a rule is ***strongly coalitional one-way monotonic*** if whenever a group of voters simultaneously all switch their rankings (which may differ) , and the effect is to switch the winning alternative from *s* alone to *t* alone, then there exists at least one voter *v* for whom $t <_\sigma s$ or $s <_\tau t$ (where σ is *v*'s initial ranking, which she switches to τ).

Proposition 4.10 Every resolute and sensible voting rule is strongly coalitional one-way monotonic.

The proof is quite similar to that of 4.6.

[18] Specifically, any scoring vector for three alternatives generates the same voting rule as some shifted version of that vector for which the middle scoring weight is 0, and any vector with middle weight 0 is a nonnegative linear combination of ⟨1, 0, 0⟩ and ⟨0, 0, -1⟩.

Example 4.11: A resolute voting rule $\mathcal{F}$ that satisfies one-way monotonicity, but not weak coalitional one-way monotonicity (and not simple monotonicity). It follows that $\mathcal{F}$ is not sensible.

We use $n = 2$ voters and $m = 3$ alternatives: a, s, and t. Consider the following rankings:

$\underline{\alpha}$	$\underline{\beta}$
a	a
t	s
s	t

Let $\mathcal{G}$ be the voting rule induced by the sensible virtue $\mathcal{V} + \mathcal{W}$, where $\mathcal{V}$ is plurality score and $\mathcal{W}$ is the following initial endowment:

$s \mapsto 2.02$ points
$t \mapsto 2.01$ points
$a \mapsto 0.00$ points

Note that $\mathcal{G}$ is resolute and one-way monotonic, and also satisfies:

1. $\mathcal{G}(\alpha\wedge\beta) = s$,
2. $\mathcal{G}(2\beta) = s$,
3. $\mathcal{G}(2\alpha) = s$,
4. $\mathcal{G}(\delta\wedge\alpha) = x$ where $\delta \notin \{\alpha, \beta\}$ and x is top-ranked in δ (so $x \in \{s, t\}$), and
5. $\mathcal{G}(\delta\wedge\beta) = x$ where $\delta \notin \{\alpha, \beta\}$ and x is top-ranked in δ (so $x \in \{s, t\}$).

Our desired rule $\mathcal{F}$ is obtained by changing $\mathcal{G}$'s value on exactly two ***exceptional*** profiles:

1. $\mathcal{F}(\alpha\wedge\beta) = a$, and
2. $\mathcal{F}(2\beta) = t$.

Note that $\mathcal{F}$ is not weakly coalitional one-way monotonic, for if both voters choose α and they simultaneously change to β then they both lift s from under t to over t, but the winner switches from s to t. Also, $\mathcal{F}$ is not simply monotonic, as $\mathcal{F}(2\alpha) = s$ and $\mathcal{F}(\alpha\wedge\beta) = a$.

It remains to prove that $\mathcal{F}$ is one-way monotonic. Consider any pivot in which the (single) pivotal voter changes from σ to τ. As there are but two voters, the profile changes from $P = \sigma\wedge\lambda$ to $Q = \tau\wedge\lambda$.

<u>Case 1</u> Assume neither P nor Q are exceptional. For the non-exceptional profiles, $\mathcal{F}$ agrees with $\mathcal{G}$, which is one-way monotonic.

<u>Case 2</u> Assume $P = 2\beta$ and $Q = \alpha\wedge\beta$, or vice versa. Then $\mathcal{F}(2\beta) = t$ and $\mathcal{F}(\alpha\wedge\beta) = a$, which does not violate one-way monotonicity.

<u>Case 3</u> Assume $P = \beta\wedge\alpha$ and $Q = 2\alpha$, or vice versa. Then $\mathcal{F}(\beta\wedge\alpha) = a$ and $\mathcal{F}(2\alpha) = s$, which does not violate one-way monotonicity.

Case 4 Assume $P = 2\beta$ and $Q = \delta\wedge\beta$ (where $\delta \notin \{\alpha, \beta\}$ and x is top-ranked in δ), or vice versa. Then $\mathcal{F}(2\beta) = t$ and $\mathcal{F}(\delta\wedge\beta) = x$, which does not violate one-way monotonicity.

Case 5 Assume $P = \alpha\wedge\beta$ and $Q = \delta\wedge\beta$ (where $\delta \notin \{\alpha, \beta\}$ and x is top-ranked in δ), or vice versa. Then $\mathcal{F}(\alpha\wedge\beta) = a$ and $\mathcal{F}(\delta\wedge\beta) = x$, which does not violate one-way monotonicity.

Case 6 Assume $P = \beta\wedge\alpha$ and $Q = \delta\wedge\alpha$ (where $\delta \notin \{\alpha, \beta\}$ and x is top-ranked in δ), or vice versa. Then $\mathcal{F}(\beta\wedge\alpha) = a$ and $\mathcal{F}(\delta\wedge\alpha) = x$, which does not violate one-way monotonicity. ∎

The example leaves open the possibility that some strong version[19] of one-way monotonicity implies sensibility.

§5 Participation and the no-show paradox

Brams and Fishburn [1983] introduced the ***no-show paradox***: one additional ***participating voter*** shows up to cast her vote, and the winner is then an alternative that is strictly inferior (according to the preferences of the participating voter) to the alternative who would have won had she not shown up. Thus, the paradox represents a type of manipulation via abstention. Moulin ([1988a] and [1988b]) expressed the corresponding form of strategy-proofness:

Definition 5.1 A (resolute) voting rule $\mathcal{F}$ for a variable electorate satisfies ***participation*** if for each profile P for a finite set N of voters, and each preference ranking σ for a participating voter v, $\mathcal{F}(P\wedge\sigma) \geq_\sigma \mathcal{F}(P)$.

By a ***variable-electorate voting rule*** $\mathcal{F}$ we mean a rule that is defined for *every* profile on *every* finite set N of voters satisfying $N \subseteq \boldsymbol{N}$, where $\boldsymbol{N}$ denotes the set of natural numbers. Moulin [1988a,b] shows that every Condorcet extension is subject to the no-show paradox. We noticed a connection to one-way monotonicity because we were obtaining some similar results, and in the next section we show that every Condorcet extension fails to satisfy one-way monotonicity. Our proof is based closely on that by Moulin, but with some elaboration that employs ideas from the proof of theorem 5.5.

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[19] It can be shown that any sensible rule $\mathcal{F}$ extends to a *social welfare rule* $\mathcal{F}^*$ that satisfies a form of one-way monotonicity appropriate to the social welfare context. (A social welfare rule yields, as the outcome of the election, a ranking of all alternatives, with ties allowed.) So any "strong version" of one-way monotonicity that implies sensibility might need to phrased in the social welfare context.

How are one-way monotonicity and participation related? A comparison requires some attention to the difference in context. Certainly one-way monotonicity *can* be considered to be a property of variable-electorate voting rules, and this is the form of the property we use in this section. But participation enforces some connection between what $\mathcal{F}$ does for profiles with n voters and what it does for profiles with $n + 1$, while one-way monotonicity does not. As we see below, this makes it easy to construct a rule that satisfies one-way monotonicity but not participation.

Such a comparison does not seem entirely fair, however – the better question may be whether one-way monotonicity implies participation in the presence of some mild axioms that do forge connections between election outcomes for different size electorates. The theorem that follows provides a positive answer, but with a strong qualification: the second axiom is not exactly "mild." It also makes use of an additional property, *half-way monotonicity*, as an interpolant between one-way monotonicity and participation. After giving the related definitions, we state the theorem.

Definition 5.2 If m is any positive integer, and P is any profile, then ***mP*** is the profile obtained from P by replacing each single voter v of P with m voters having the same preference as v. An anonymous, variable-electorate voting rule $\mathcal{F}$ is ***homogeneous*** if $\mathcal{F}(mP) = \mathcal{F}(P)$ holds for all choices of P and m.[20]

Definition 5.3 For any any (strict) ranking σ, let ***rev(σ)*** denoting the ranking obtained by reversing σ (so that $x <_{\sigma} y$ iff $y <_{rev(\sigma)} x$). For a profile P and a strict ranking σ let $P \wedge \sigma \wedge rev(\sigma)$ denote the profile obtained by adding one additional voter with ranking σ and a second additional voter with ranking $rev(\sigma)$. An anonymous, variable-electorate voting rule $\mathcal{F}$ satisfies ***reversal cancellation*** if for all choices of P and σ, $\mathcal{F}(P) = \mathcal{F}(P \wedge \sigma \wedge rev(\sigma))$.[21]

Definition 5.4 A resolute voting rule $\mathcal{F}$ is ***half-way monotonic*** if for every pivot $(P, v, \sigma \to s, rev(\sigma) \to t)$ for $\mathcal{F}$, $s >_{\sigma} t$.

Half-way monotonicity requires that whenever the effect of a pivotal voter's total reversal in vote from σ to rev(σ) is to switch the winner of the election from s to t, <u>the pivotal</u>

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[20] Homogeneity is a very weak form of ***reinforcement*** (also known as ***consistency***), discussed in JH Smith [1973]. It is known to hold for almost every social choice rule, though Fishburn [1977] shows that the Dodgson and Young procedures can each fail to be homogeneous, depending on the some details in the precise formulation of these systems. It is probably fair to deem homogeneity an "innocuous" assumption.

[21] Reversal cancellation is closely related to work by Saari [1994], [1999], and by Saari and Barney [2004], who consider the effect of reversing an entire profile, and examines the vector component of a profile corresponding to reversal. Every net pairwise rule (see §2) satisfies reversal cancellation; these include Borda, Copeland, and Simpson. But the property fails for other scoring rules, such as plurality and antiplurality (and fails for Hare) so it can hardly be called innocuous.

voter must have raised t from below s according to σ, to above s according to rev(σ). Notice that the following three requirements are each equivalent to the underlined phrase:

- $s >_{\sigma} t$
- $s >_{\sigma} t$ or $t >_{rev(\sigma)} s$
- $s >_{\sigma} t$ and $t >_{rev(\sigma)} s$

One consequence is that if we were to start with our stronger property of two-way monotonicity (with the "and") and weaken it by requiring that $\tau = rev(\sigma)$, we would get the same property – half-way monotonicity – as we do when we similarly weaken one-way monotonicity.

Note that half-way monotonicity has an interesting interpretation in terms of strategy-proofness. A rule that fails to be half-way monotonic can be manipulated by some voter who *completely* misrepresents her preferences, in the sense that she announces a preference ranking that misstates every possible pairwise comparison among alternatives.

Theorem 5.5 Consider properties of resolute, variable-electorate voting rules. Then

(i) one-way monotonicity and participation are independent,
(ii) participation ⇒ half-way monotonicity,
(iii) [half-way monotonicity + homogeneity + reversal-cancellation] ⇒ participation,
(iv) for the case of exactly three alternatives, participation ⇒ one-way monotonicity, and
(v) for the case of exactly four alternatives, [participation + simple monotonicity] ⇒ one-way monotonicity.

Corollary 5.6 (immediate) For resolute, anonymous, variable-electorate voting rules satisfying both homogeneity and reversal cancellation:

- Participation and half-way monotonicity are equivalent. In other words, participation is equivalent to the corresponding weak form of strategy-proofness stating that no one can improve the outcome through such a complete misrepresentation.
- Participation, half-way monotonicity, and one-way monotonicity are all equivalent for the special case of three alternatives, or of four alternatives with the additional assumption of simple monotonicity.

We give the long proof of theorem 5.5 at the end of the section.

In the presence of homogeneity + reversal-cancellation, might one-way monotonicity and participation be equivalent? Our proof of part (i) leaves this possibility open, but we conjecture that one-way monotonicity is strictly stronger than participation with these assumptions.

Our proof of theorem 5.5 exploits an additional similarity between one-way monotonicity and participation – there exists a notion of ***variable electorate virtue*** with some properties analogous to those of section 3. Such a virtue $\mathcal{V}$ is defined for a variable electorate, and is ***participation sensible*** if each profile P and ranking σ satisfies $x \geq_{\sigma} y \Rightarrow \Delta x_{\mathcal{V}} \geq \Delta y_{\mathcal{V}}$, for every two alternatives x and y. Here, $\Delta x_{\mathcal{V}}$ denotes $\mathcal{V}_{P\wedge\sigma}(x)$ - $\mathcal{V}_P(x)$. The following proposition is now easy to verify, and we leave the proof to the reader.

Proposition 5.7 The participation axiom is satisfied by any variable-electorate, resolute voting rule that is induced by a participation sensible variable electorate virtue.

We now turn to the proof of theorem 5.5.

Proof of theorem 5.5

Part (i) The following variable electorate voting rule satisfies one-way monotonicity but not participation: Let $\mathcal{F}$ act as the plurality rule$^{\prec}$ for profiles with *5* or fewer voters, and as the Borda count$^{\prec}$ with *6* or more. Then $\mathcal{F}$ is one-way monotonic by Corollary 4.7. To see that participation fails, consider the *5*-voter profile below, and a participating voter with ranking σ:

P						σ
1	*1*	*1*	*1*	*1*		
s	*s*	*t*	*x*	*y*		*x*
t	*t*	*y*	*t*	*t*		*y*
x	*y*	*x*	*y*	*x*		*s*
y	*x*	*s*	*s*	*s*		*t*

Note that $\mathcal{F}(P) = s$, while $\mathcal{F}(P\wedge\sigma) = t$.

The following rule $\mathcal{G}$ for four alternatives satisfies participation, but not one-way monotonicity. For the record, we note that $\mathcal{G}$ is anonymous, but does not satisfy neutrality, homogeneity, or reversal cancellation. As predicted by part (v), $\mathcal{G}$ also fails to be simply monotonic. Certainly it would be interesting to find a similar example with five or more alternatives, in which simple monotonicity holds.

Description of $\mathcal{G}$

1) Our four alternatives are *x*, *y*, *s*, and *t*.
2) Our preliminary version of $\mathcal{G}$ is the scoring rule with scoring weights *3*, *2*, *2*, *1*, but this is modified by the remaining clauses.
3) Each alternative has a fixed endowment of points before the voting begins, which is added to that alternative's point total, as determined by the profile at hand, to determine that alternatives ***final score***. The endowments are as follows:

   $x \mapsto 2.03$ *points*

   $y \mapsto 0.02$ *points*

   $s \mapsto 2.01$ *points*

   $t \mapsto 2.0$ *points*

The effect is the same as giving *x*, *s*, and *t* each *2* points, with *0* points to *y*, and imposing the tie-breaking agenda $x \succ y \succ s \succ t$ on the penultimate outcome.

4) There is a single profile, which we will refer to as "exceptional," to which the above rules do not apply. It is the profile P_β in which there is exactly one voter, who has the following preference ranking β:

β	
y	*3.02*
s	*4.01*
t	*4.0*
x	*3.03*

The numbers in the right column give the final scores, so that the fractional amounts, in effect, break the tie in favor of s. Instead, we declare that $\mathcal{G}(P_\beta) = t$.

Proof that $\mathcal{G}$ satisfies participation, but fails both simple monotonicity and one-way monotonicity

Consider the one-voter profile in which the single voter has the following ranking α:

α	
y	*3.02*
t	*4.0*
s	*4.01*
x	*3.03*

This is not the exceptional profile, and so $\mathcal{G}(P_\alpha) = s$. The pivot $(P_\alpha, v, \alpha \rightarrow s, \beta \rightarrow t)$ shows that $\mathcal{G}$ fails both simple monotonicity and one-way monotonicity. To see that $\mathcal{G}$ satisfies participation consider the transition from some profile Q to the profile $Q \wedge \sigma$, where σ is the ranking of the newly participating voter.

Case 1 Assume neither Q nor $Q \wedge \sigma$ is P_β. Then participation holds for this transition, as $\mathcal{G}$ is completely given by clauses 1) - 3), which describe a participation-sensible virtue.

Case 2 Assume $Q \wedge \sigma = P_\beta$. Then Q is the "empty profile" with no voters, $\mathcal{G}(Q) = x$ and $\mathcal{G}(Q \wedge \sigma) = t$. The participating voter has ranking β, which ranks *t* over *x*, so participation is satisfied for this transition.

Case 3 Assume $Q = P_\beta$. We claim that $\mathcal{G}(Q \wedge \sigma)$ is equal to the top-ranked alternative of σ. The claim suffices to show that participation is satisfied for this transition.

Subcase 3.1 Assume that *s* is top-ranked by σ. By referring to the final scores for P_β we see that the final score for *s* in profile $P_\beta \wedge \sigma$ is *7.01*, which is greater than any other final score, so that $\mathcal{G}(P_\beta \wedge \sigma) = s$.

Subcase 3.2 Assume that *t* is top-ranked by σ. The argument is similar to that of the previous subcase.

Subcase 3.3 Assume that y is top-ranked by σ. Then the final score for y in profile $P_\beta \wedge \sigma$ is *6.02* and for x is at most *5.03*; s and t each get at most *6.01*, and $\mathcal{G}(P_\beta \wedge \sigma) = y$.

Subcase 3.4 Assume that x is top-ranked by σ. Then the final score for x in profile $P_\beta \wedge \sigma$ is *6.03*; each other alternative gets at most *6.02*, and $\mathcal{G}(P_\beta \wedge \sigma) = x$.

Parts (ii), (iv) and (v) Our somewhat unorthodox approach will be to launch an attempt to prove that participation implies one-way monotonicity. This proof will break down in one of the cases. We will then observe that the obstacle is circumvented under the additional assumption that $\tau = rev(\sigma)$, or that there are only three alternatives. Alternatively, the obstacle is circumvented for exactly four alternatives if the rule is simply monotonic. This approach suggests some insight into the relationship between the four properties under consideration.

Assume that $\mathcal{F}$ satisfies participation, and that $(P, v, \sigma \to s, \tau \to t)$ is a pivot for $\mathcal{F}$. That is, $\mathcal{F}(P \wedge \sigma) = s$, and $\mathcal{F}(P \wedge \tau) = t$. We must show that $s >_\sigma t$ or $t >_\tau s$.

Case 1 Assume that $\mathcal{F}(P) = s$ or $\mathcal{F}(P) = t$. If $\mathcal{F}(P) = s$ then as $\mathcal{F}(P \wedge \tau) = t$, $t >_\tau s$ by participation. If $\mathcal{F}(P) = t$, then as $\mathcal{F}(P \wedge \sigma) = s$, participation implies $s >_\sigma t$.

Case 2 Assume that $\mathcal{F}(P \wedge \sigma \wedge \tau) = s$ or $\mathcal{F}(P \wedge \sigma \wedge \tau) = t$. If $\mathcal{F}(P \wedge \sigma \wedge \tau) = s$, then as $\mathcal{F}(P \wedge \tau) = t$, participation implies $s >_\sigma t$. If $\mathcal{F}(P \wedge \sigma \wedge \tau) = t$ then as $\mathcal{F}(P \wedge \sigma) = s$, participation implies $t >_\tau s$.

Case 3 Assume that $\mathcal{F}(P) = x$ with $x \notin \{s, t\}$ and $\mathcal{F}(P \wedge \sigma \wedge \tau) = y$ with $y \notin \{s, t\}$. As $\mathcal{F}(P) = x$ and $\mathcal{F}(P \wedge \sigma) = s$, participation implies $s >_\sigma x$. Also, from $\mathcal{F}(P) = x$ and $\mathcal{F}(P \wedge \tau) = t$, participation implies $t >_\tau x$. But from $s >_\sigma x$ and $t >_\tau x$ we can draw no conclusion about how σ or τ rank s versus t. However, if $\tau = rev(\sigma)$ then "$s >_\sigma x$ and $t >_\tau x$" becomes "$s >_\sigma x$ and $t >_{rev(\sigma)} x$," which is "$s >_\sigma x$ and "$x >_\sigma t$," whence $s >_\sigma t$, as desired. Similarly, from $\mathcal{F}(P \wedge \sigma \wedge \tau) = y$ and $\mathcal{F}(P \wedge \sigma) = s$, participation implies $y >_\tau s$. Also from $\mathcal{F}(P \wedge \sigma \wedge \tau) = y$ and $\mathcal{F}(P \wedge \tau) = t$, participation implies $y >_\sigma t$. Again, we can conclude nothing from "$y >_\tau s$ and $y >_\sigma t$," unless $\tau = rev(\sigma)$, in which *case* $s >_\sigma t$ again follows. Finally, observe that we in fact have *four* facts to work with:

- $s >_\sigma x$,
- $t >_\tau x$,
- $y >_\tau s$, and
- $y >_\sigma t$

If we knew $x = y$, then we could conclude both $s >_\sigma t$ and $t >_\tau s$ with no additional assumption that $\tau = rev(\sigma)$. If there are exactly three alternatives, then the case 3 assumption implies $x = y$, and we conclude that $\mathcal{F}$ is one-way monotonic. If there are exactly four alternatives s, t, x, and y, then the four inequalities just listed, coupled with the assumption that one-way monotonicity fails (in that $t >_\sigma s$ and $s >_\tau t$), completely determines the orderings σ and τ: $y >_\sigma t >_\sigma s >_\sigma x$, and $y >_\tau s >_\tau t >_\tau x$. As $\mathcal{F}(P\wedge\sigma) = s$, and $\mathcal{F}(P\wedge\tau) = t$, these orderings yield a failure of simple monotonicity. Thus, with exactly four alternatives, if $\mathcal{F}$ satisfies participation, then any failure of one-way monotonicity implies a failure of simple monotonicity.

Part (iii) Consider a failure of participation: a profile P and a single added voter v with ranking σ such that $t = \mathcal{F}(P\wedge\sigma)$ is below $s = \mathcal{F}(P)$ according to the ranking σ. That is, $\mathcal{F}(P\wedge\sigma) <_\sigma \mathcal{F}(P)$. Then $\mathcal{F}((2P)\wedge\sigma\wedge rev(\sigma)) = \mathcal{F}(2P) = \mathcal{F}(P) = s$, and $\mathcal{F}((2P)\wedge\sigma\wedge\sigma) = \mathcal{F}(2(P\wedge\sigma)) = \mathcal{F}(P\wedge\sigma) = t$. But the profile $(2P)\wedge\sigma\wedge\sigma$ is obtained from the profile $(2P)\wedge\sigma\wedge rev(\sigma)$ by having the voter v with preference ranking $rev(\sigma)$ flip his ranking upside down so that it becomes σ. This voter raises s from below t in $rev(\sigma)$ to above t in σ, and the effect is that s now loses while t wins . . . a failure of half-way monotonicity. ∎

From theorem 5.5 together with Moulin's result that every Condorcet extension fails participation, we can immediately conclude that every homogeneous and reversal-canceling Condorcet extension fails one-way monotonicity. However, our direct modification of Moulin's proof in the next section avoids the need to assume homogeneity and reversal cancellation.

§6 Negative results: Condorcet extensions, Hare, and plurality run-off

Our approach to the main result makes use of the following coalitional version of half-way monotonicity:

Definition 6.1 A resolute voting rule is ***weakly coalitional half-way monotonic*** if whenever a set of voters having *identical* strict ranking σ all simultaneously change their votes to $rev(\sigma)$, and the effect is to switch the winner from s alone to t alone, it must be that $s >_\sigma t$.

Proposition 6.2 Half-way monotonicity implies weak coalitional half-way monotonicity.[22]

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[22] The proof that follows applies to any ***path connected*** domain – any domain $\mathcal{D}$ with the property that every pair of profiles in $\mathcal{D}$ is linked by some ordered chain of "connecting" profiles such that each profile differs in only one voter from the next in the chain. Note that the pathwise connected domains include all domains obtained as some cartesian product of restricted sets of preference rankings. One can formulate a strong coalitional version of half-way monotonicity, as well as weak and strong coalitional versions of participation, simple monotonicity, and two-way monotonicity (with strong coalitional two-way monotonicity equivalent to coalitional strategy-proofness). It then turns out that there is considerable variation as to whether the simple iteration argument used in the proof of 6.2 suffices to derive one or

Proof Assume a set of k voters having identical strict ranking σ all change their votes to $rev(\sigma)$, and the effect is to switch the winner from s to t. Let P_j be the profile in which j of these k voters have changed from σ to $rev(\sigma)$, and $k - j$ have not changed, and consider the sequence of profiles $P = P_0, P_1, \ldots, P_k$. Suppose the winners for the profiles $P_0, P_1, \ldots, P_k$ are $s = x_0, x_1, \ldots, x_u = t$ and that x_q is the (unique) winner for profiles $P_{j(q)}, P_{j(q)+1}, \ldots, P_{j(q+1)-1}$, with $0 = j(1) < j(2) < \ldots < j(u) \le t$. Then by ordinary half-way monotonicity applied to the transition from $P_{j(q+1)-1}$ to $P_{j(q+1)}$, for $r = 0, 1, \ldots, u - 1$, it follows that $s >_\sigma x_1 >_\sigma \ldots >_\sigma x_u = t$, so that $s >_\sigma t$, as desired. ∎

Theorem 6.3 With four or more alternatives and sufficiently many voters, no Condorcet extension satisfies *weak coalitional half-way monotonicity*.

Corollary 6.4 With four or more alternatives and sufficiently many voters, no Condorcet extension satisfies half-way monotonicity, and so no Condorcet extension satisfies one-way monotonicity.

Proof of theorem 6.3 Our proof is an elaborated version of the argument in Exercise 9.3(c), page 251 of Moulin [1988a]. The elaboration uses the ideas behind the homogeneity and reversal cancellation axioms, but does not require any assumption that these axioms hold. (Rather, it uses that these axioms hold, speaking loosely, for Condorcet winners and for Simpson scores.) Let C denote the profile, for $m \ge 4$ alternatives, in which each possible ranking occurs exactly once. Note C has $m!$ voters. For an arbitrary profile P with $n = \mathbf{n(P)}$ voters, let $k = \mathbf{k(P)}$ be the maximum integer j such that each of the $m!$ rankings occurs at least j times in P. Informally, $k(P)$ represents the "number of copies of C contained in P." Let $\mathbf{n^*(P)}$ denote $n(P) - (m!)k$. Informally, n*(P) is the number of voters remaining once one ignores the copies of C.

Definition 6.5 ***Condition M*** holds of profile P *if* $2k(P) \ge n^*(P) + 2$.

Informally, condition M says that P contains "enough" copies of C relative to the number of voters who would remain if all copies of C were removed.

Claim Let $\mathcal{F}$ be a Condorcet extension satisfying weak coalitional half-way monotonicity. Let a and b be any two alternatives, and P be any profile for four or more alternatives that meets the following three conditions:

- condition M,
- the net Simpson score $t^*(b)$ is even and $t^*(b) \le 0$, and
- $Net(b > a) > -t^*(b) + 2$.

Then $\mathcal{F}$ does not elect alternative a at profile P.

both coalitional forms from the individual form. In particular, this argument does show that the strong coalitional forms of Maskin monotonicity (which is, in fact, the standard form in the literature) and of simple monotonicity follow respectively from the individual forms we have defined here (see footnotes 10 and 11) for every path connected domain. On the other hand, Example 4.11 shows that the weak coalitional form of one-way monotonicity does not follow from the individual form.

Proof of claim Let a, b, and P be as stated. Note that in general, we know that $t^*(b)$ satisfies $-n \leq t^*(b) \leq n$. However, copies of profile C have no effect on the value of $t^*(b)$, so in fact we know $-n^* \leq t^*(b) \leq n^*$, whence $-t^*(b) + 2 \leq n^* + 2 \leq 2k$.

Let $r = \frac{-t^*(b)+2}{2}$, a strictly positive integer. Choose σ to be any ranking such that a is at the bottom, b is immediately above a, and all other alternatives are ranked above b. By assumption M there exist at least r voters who voted σ. Now let Q be obtained from P by having r voters who voted σ all change their votes to $rev(\sigma)$. For each alternative x other than a or b, the effect of these changes is that

$$Net_Q(b > x) = Net_P(b > x) + 2r = Net_P(b > x) + -t^*(b) + 2.$$

As $Net_P(b > x) \geq t^*(b)$, this makes $Net_Q(b > x) > 0$. Furthermore

$$Net_Q(b > a) = Net_P(b > a) - 2r = Net_P(b > a) + t^*(b) - 2 > 0.$$

Hence, b is a Condorcet winner for profile Q (and a is not a winner). If a had been a winner for P, this would be a violation of weak coalitional half-way monotonicity. This completes the proof of the claim.

Now consider the following profile R:

6	6	10	8
a	a	d	b
d	d	b	c
c	b	c	a
b	c	a	d

with $n(R) = n^*(R) = 30$. Let P be obtained from R by adding 28 copies of the profile C. Then $n^*(P) = n^*(R) = 30$.

The claim can now be applied three times to show that b, c, and d cannot be elected at profile P, so that a is the sole winner. (Note that when calculating any value of t^* or $Net(x > y)$, C can be ignored, so the values for P are the same as those for R.) Next suppose that four of the voters from P who have ranking $d > b > a > c$, simultaneously reverse their rankings, yielding some new profile Q. Note that $n^*(Q) = 30 + 8 = 38$ and $k(Q) = k(P) - 8 = 20$, so that Q satisfies condition M. Apply the claim two more times to show that neither a nor c can be elected at Q. This contradicts our assumption that $\mathcal{F}$ is weakly coalitional half-way monotonic. ∎

The situation painted by 6.3 seems somewhat odd. On the one hand, it is easy to see that on the domain $\mathcal{D}_{Condorcet}$ of profiles having Condorcet winners, pairwise majority rule satisfies the strong property of two-way monotonicity. But it is impossible to extend pairwise-majority rule over the full domain without violating the weaker property of one-way monotonicity. Meanwhile, there are rules such as scoring rules or the omninominator rule that "do less well" than does pairwise majority rule on $\mathcal{D}_{Condorcet}$, yet do better on the full domain. It is almost as if pairwise majority rule paints itself into a corner by trying too hard on $\mathcal{D}_{Condorcet}$.

Next, we consider two closely related voting rules.

Definition 6.7 In ***plurality with run-off***, if no candidate achieves a strict majority of first-place votes, there is a run-off between the two alternatives x and y achieving the greatest number of first-place votes: the winning alternative is whichever of x or y is ranked over the other by a majority of voters. In the ***Hare*** rule (or "alternative vote," as it termed in Moulin [1988a]) alternatives are eliminated in sequential stages, based on fewest first-place votes. Each stage considers only the relative rankings over surviving alternatives, and the winner is the last alternative (or final group of tied alternatives) remaining.

The following profile R is adapted from one in Moulin [1988a] to show that neither of these two rules are simply monotonic:

6	5	6	
a	*c*	*b*	
b	*a*	*c*	Profile R
c	*b*	*a*	

We will reason about both plurality run-off and Hare together. In the above profile R, c is eliminated, all *5* votes for c then turn to a, who wins the run-off against b. However, if two of group of *6* who ranked b on top change rankings as indicated below

6	5	4	2	
a	*c*	*b*	*a*	
b	*a*	*c*	*b*	Profile R''
c	*b*	*a*	*c*	

Then in R'' it is b who is eliminated and c wins the run-off against a. This represents a failure of both simple monotonicity and weak coalitional one-way monotonicity – but can we obtain a failure of ordinary one-way monotonicity? It makes sense to consider the intermediate profile below, in which only one of the six b-voters has made the switch:

6	5	5	1	
a	*c*	*b*	*a*	
b	*a*	*c*	*b*	Profile R'
c	*b*	*a*	*c*	

The most straightforward interpretation of "Plurality with run-off" seems to be that for this profile both alternatives b and c would be eliminated, leaving a the winner. In that case, the transition from R' to R'' provides the desired failure of one-way monotonicity. This interpretation appears to be a standard one for the Hare (alternative vote), so we conclude that Hare fails one-way monotonicity. What else might plurality run-off actually would do in a situation such as R'? We imagine that the ambiguity may not be of much concern, at least not for large presidential elections in which exact ties are extremely unlikely (or even ill-defined, as truly exact vote counts do not seem to exist in the real world).

The (only) other alternative that suggests itself is that plurality run-off might declare a three-way tie among a, b, and c for R'. In that case, if the tie-breaking dictator to throw the contest to a, then the transition from R' to R'' again provides the desired failure of one-way monotonicity. This is not an entirely satisfactory solution, so we will phrase the corresponding proposition conservatively:

Proposition 6.8 The Hare rule fails to be one-way monotonic. Plurality run-off fails to satisfy weak coalitional one-way monotonicity.

§7 Conclusions

One-way monotonicity stands apart from previously studied monotonicity properties because of its distinct interpretation in terms of strategy-proofness. This interpretation suggests the possibility of new approaches to the problem of measuring the relative degree of strategy-proofness for standard voting rules. Our feeling is that one-way monotonicity has some additional normative appeal, apart from this interpretation, and discriminates in a useful way among realistic voting rules, so that it has some features common to each of the classes we described in the introduction.

At the same time, one-way monotonicity shares important qualitative features with the participation axiom, in terms both of shared negative results for Condorcet extensions, and of positive results for voting rules induced by certain types of cardinal functions, called here sensible virtues, that respond appropriately to changes in a profile. We do not yet understand the exact relationship between one-way monotonicity and sensible virtues, but it seems that the strong coalitional form, and the social welfare form, of one-way monotonicity play a role. The same comments apply to participation.

In terms of logical strength, there seems to be an intricate relationship among one-way monotonicity, half-way monotonicity, participation, simple monotonicity, and some other axioms that bridge the gap between properties for a fixed electorate and those for a variable electorate. In one sense, theorem 5.5 together with the counterexamples provided in Campbell and Kelly [2002] and in this paper, already tell us a lot about these relationships.

These counterexamples, however, typically fail to be neutral or fail to be anonymous, and so they do not address questions such as the following:

(*) ***Does participation imply one-way monotonicity for neutral and anonymous rules?***

This question may at first seem to be poorly conceived, as neutral and anonymous rules would need to be rendered resolute before the question made sense, and the mechanism employed (such as a tie-breaking agenda) would destroy one or the other of neutrality and anonymity. The question would become

(**) ***Does participation imply one-way monotonicity for neutral and anonymous rules, after they are rendered resolute via a tie-breaking agenda?***

which may not seem to be all that interesting.

On the other hand, one might approach question (∗) in a different way, by adapting the properties directly so that they make sense when applied to irresolute voting rules:

(∗∗∗) *Does the irresolute form of participation imply the irresolute form of one-way monotonicity, for neutral and anonymous rules?*

Here is one possible way to extend one-way monotonicity to the irresolute context. We define an ***irresolute pivot*** for a voting rule $\mathcal{F}$ to be a vector $\Theta = (P, v, \sigma \rightarrow \{s...\}, \tau \rightarrow \{t...\})$ satisfying:

- $v \in N$ is a voter and P is a profile for the set $N - \{v\}$,
- σ and τ are strict rankings of A, and s and t are alternatives in A,
- $s \in \mathcal{F}(P \wedge \sigma)$,
- $t \in \mathcal{F}(P \wedge \tau)$, and
- either $t \notin \mathcal{F}(P \wedge \sigma)$ or $s \notin \mathcal{F}(P \wedge \tau)$.

An irresolute voting rule $\mathcal{F}$ is then said to be ***irresolutely one-way monotonic*** if for every irresolute pivot $(P, v, \sigma \rightarrow \{s...\}, \tau \rightarrow \{t...\})$ for $\mathcal{F}$, $t <_\sigma s$ or $s <_\tau t$. In Sanver and Zwicker [2007], we show that an irresolute voting rule $\mathcal{F}$ is irresolutely one-way monotonic in this sense if and only if for *every* choice of a tie-breaking agenda $\prec$, the resolute rule $\mathcal{F}^{\prec}$ is one-way monotonic according to the definition we have used throughout this paper.

One implication of this result is that the method we have used in this paper, of rendering voting rules resolute via a tie-breaking agenda, is less problematic than may first appear (see discussion in §2, including footnote 7). Another message is that question (∗∗) is more natural than one might think; in effect, it is identical to an instance of question (∗∗∗).

The most common approach for adapting *strategy-proofness* to the irresolute context is to extend, in any one of a number of possible ways, a voter's preferences over individual alternatives to preferences over sets of alternatives (see, for example, Gärdenfors [1979], or Taylor [2005]). This method may be applied to monotonicity properties, as well, suggesting an alternative to the approach sketched above. In Sanver and Zwicker [2007] we compare several alternative approaches for generalizing one-way and two-way monotonicity, simple monotonicity, and participation to the irresolute context. In general, we find (perhaps unsurprisingly) that the study of irresolute monotonicity is rich and complex. However, some of our results suggest that there may be enough agreement among the various approaches to avoid a devolution into Byzantine intricacy.

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