CHAPTER 0: BASIC MATHEMATICS

0.1 BASIC LOGIC

Before studying any topic in mathematics, one should be very comfortable with certain basic notions of logic. We will go over these in this section. Although this section is simple, it contains the basics of reasoning, without which there are no mathematics. So make sure that everything in this section is very well understood.

We will start with the very important notion of necessary and sufficient conditions. Let’s explain this by examples. Assume I say the following:

You succeed if you study

This means the following: whenever you study you succeed. Thus studying implies success and we denote this as:

studying \implies \text{success}

Note that here studying is a sufficient condition for success. Whenever you study, you succeed. But note that my above statement does not say that there is no other way to succeed. So in this statement studying is not necessary to succeed.

Assume I had told the following:

You succeed only if you study

This is very different than the above statement. I say here that the only way of success is studying. Here success implies studying and we denote this as:
So studying is a necessary condition for success. Note that this statement does not say that studying guarantees success, it is only a must for success. So here studying is not a sufficient condition to succeed.

Remark that if we have an implication such as

\[ \text{success} \implies \text{studying} \]

the equivalent of this is the implication

\[ \neg \text{studying} \implies \neg \text{success}. \]

Note that

\[ \neg \text{success} \implies \neg \text{studying} \]

is the same as

\[ \text{studying} \implies \text{success} \]

and definitely not the same as

\[ \text{success} \implies \text{studying} \]

Finally I could have told the following:

You succeed if and only if you study

Here studying is both a necessary and sufficient condition for success. We denote this as

\[ \text{studying} \iff \text{success} \]
We call this a characterization. If two concepts are connected with an “if and only if” statement, they turn out to be equivalent. Here success is equivalent to studying. Whenever you study, you succeed; whenever you succeed this means that you study. If you don’t study you don’t succeed; if you don’t succeed this means that you don’t study.

Now, we will turn to the idea of a quantifier. Let’s illustrate this by an example:

Consider the set of all apples that we denote by $A$. Consider also the claim that an object is red. Let us give a name to this claim, e.g., $p$. So we have

$$p: \text{being red}$$

We can apply this claim to all of the elements of the set $A$. In this case we will be claiming that all elements of $A$ are red, i.e. all apples are red. To express this, we use the quantifier “for all” that we denote by $\forall$.

So if we write

$$\forall a \in A \text{ we have } p$$

this should be read as “for any element of the set $A$, the claim $p$ holds”. In plain English this says all apples are red.

Of course we may have this claim not for all elements of $A$ but only for some elements of it. We may want to say that there exist apples which are red, which is totally different than saying that all apples are red. The quantifier that we use in this case is “there exists”, which we denote by $\exists$.

So if we write

$$\exists a \in A \text{ such that } p$$
this should be read as “there is an element of the set \( A \) (possibly more than one), for which the claim \( p \) holds”. In plain English this says that there exists at least one red apple.

By a *proposition*, we mean any claim about any object, which can be classified as either true or false. *Proving a proposition* is to show that the proposition is true, and *negating a proposition* is to show that it is false.

We use different methods for proving and negating propositions, depending on their quantifiers. If we have to prove a proposition with the quantifier “for all”, we have to show that our claim is valid for all of the elements of the set under analysis. For example, if we want to prove the proposition “all apples are red”, we have to show that the claim of being red holds for all elements of the set of apples. It is not sufficient to give only an example of a red apple.

In contrast, if we want to prove the proposition “there exists a red apple”, giving an example of a red apple will be sufficient.

If we have to negate a proposition with the quantifier “for all”, we don’t have to show that our claim is false for all of the elements of the set under analysis. It is sufficient to give a single example where the claim does not hold. If we want to negate the proposition “all apples are red”, it is sufficient to show an apple which is not red. In contrast, if we want to negate the proposition “there exists a red apple”, it is not sufficient to give an example of an apple which is not red, but we have to show that the claim of being red is false for all elements of the set of apples.

Remark that if a proposition \( p \) is false then its negation “not \( p \)” that we denote by \( p' \) is true.

Finally we want to note that propositions can be connected two-by-two (i.e., binarily), with each other. This brings us to the idea of binary operators. We will consider three binary operators: and, or, either-or.
If we say that an apple is red and round, this means that the apple has both of the properties of being red and being round.

If we say that an apple is red or round, this means that the apple has at least one of the properties of being red and being round. It may be red but not round; round but not red; both round and red. This is what we call an “inclusive or”. In contrast, if we say that an apple is either red or round, this is what we call an “exclusive or”: it implies that an apple has one and only one of the properties of being red and being round: an apple may be red but not round; round but not red. It cannot be red and round at the same time.
0.2 SET THEORY

A set is a collection of objects. Now we will learn two ways of defining sets.

The first way to define a set is to write its elements explicitly within braces. For example:

\{1, 2, 3\}

is the set whose elements are the numbers 1, 2 and 3. It is clear that only sets which have a small number of elements can be written in this way. In many cases we define a set, by characterizing its elements by a certain property. Consider, for example, the following set:

\{x \in \mathbb{R} : x > 0\}

Here \(\mathbb{R}\) is the set of real numbers and the symbol "\(:\)" means "such that". Thus the above expression can be read as follows: "any \(x\) which is an element of the set of real numbers such that \(x\) satisfies the property of being strictly positive. So the set \(\{x \in \mathbb{R} : x > 0\}\) is the set of all real numbers which are strictly positive.

We will now give certain basic definitions about sets.

Definition 0.2.1: The cardinality of a set \(S\) is the number of elements it contains. We denote it as \(|S|\).

For example if \(S = \{a, b, c, d\}\), we have \(|S| = 4\).

A set \(S\) is said to be finite (resp., infinite) whenever \(|S|\) is finite (resp., infinite).

Definition 0.2.2: We say that a set \(S\) is a subset of a set \(T\) if every element of the set \(S\) is also an element of the set \(T\).

We denote this as \(S \subseteq T\).
Note that we could have given the above definition as follows:

**Definition 0.2.3:** We say that a set $S$ is a subset of a set $T$ if $\forall x \in S$ we have $x \in T$.

You should understand that Definition 0.2.2 and Definition 0.2.3 are the same.

**Remark 0.2.1:** Note that according to Definition 0.2.2 every set is a subset of itself. Moreover, the empty-set is a subset of all sets.

We will now define the notion of a proper subset.

**Definition 0.2.4:** We say that a set $S$ is a proper subset of a set $T$ if every element of the set $S$ is also an element of the set $T$ and there are elements of $T$ which are not elements of $S$.

We denote this as $S \subset T$.

Note that we could have given the above definition as follows:

**Definition 0.2.5:** We say that a set $S$ is a proper subset of a set $T$ if $\forall x \in S$ we have $x \in T$ and $\exists y \in T$ such that $y \not\in S$.

You should again understand that Definition 0.2.4 and Definition 0.2.5 are the same.

**Definition 0.2.6:** We say that two sets $S$ and $T$ are equal if and only if we have $S \subset T$ and $T \subset S$. We denote this as $S = T$.

Now we will define three set operations: union, intersection and difference of sets.

- The union of two sets $S$ and $T$ is denoted as $S \cup T$, and it is defined as $S \cup T = \{x : x \in S \text{ or } x \in T\}$. Remark that the operator “or” that we use here is an “inclusive or”.
• The intersection of two sets $S$ and $T$ is denoted as $S \cap T$, and it is defined as $S \cap T = \{ x : x \in S \text{ and } x \in T \}$.

• The difference between two sets $S$ and $T$ is denoted as $S \setminus T$, and it is defined as $S \setminus T = \{ x : x \in S \text{ and } x \notin T \}$.

We finally define the notion of a power set.

**Definition 0.2.7:** The *power set* of a set $S$ is the set of all its subsets. We will denote it as $2^S$.

**Example 0.2.1:** If $S = \{a, b, c\}$ then $2^S = \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \}$

**Remark 0.2.2:** Any element of $2^S$ is a subset of $S$. For example we have $\{a\} \in 2^S$ but not $a \notin 2^S$. In contrast $a \in S$.

We will finally state a result about the cardinalities of power sets.

**Theorem 0.2.1:** Consider any finite set $S$ with $|S| = n$. We then have $|2^S| = 2^n$.

**Proof:** omitted
0.3 BINARY RELATIONS

We will first start by defining what we mean by a Cartesian product of two sets.

**Definition 0.3.1:** Given any two sets \( S \) and \( T \), their *Cartesian product* is defined as the set \( S \times T = \{(s,t) : s \in S \text{ and } t \in T\} \)

**Example 0.3.1:** If \( S = \{s_1, s_2\} \) and \( T = \{ t_1, t_2, t_3\} \), then \( S \times T = \{(s_1, t_1), (s_1, t_2), (s_1, t_3), (s_2, t_1), (s_2, t_2), (s_2, t_3)\} \)

An element of \( S \times T \) is called an *ordered pair*.

**Remark 0.3.1:** \( S \times T \neq S \times T \)

**Remark 0.3.2:** We can also generalize the definition of an ordered pair to ordered triples, quadruples, etc. So we can define the Cartesian product of three, four, brief any number of sets.

In mathematics we often explore relationships between objects (numbers, sets, functions, etc.). If an element \( s \) of a set \( S \) is related to an element \( t \) of a set \( T \) by a relation \( R \) we may write “\( s \) is related to \( t \)”, or shortly “\( sRt \)”, or even more shortly \((s,t)\). This leads us to define the notion of a relation between two sets.

**Definition 0.3.2:** Given any two sets \( S \) and \( T \), a *relation* \( R \) from \( S \) into \( T \) is a subset of \( S \times T \).

**Example 0.3.2:** \( \{(x, y) \in \mathbb{Z} \times \mathbb{N} : |x| = y\} \) is a relation from the set of integers into the set of natural numbers.

**Definition 0.3.3:** Given a set \( S \), a *binary relation* \( R \) on \( S \) is a subset of \( S \times S \).
Example 0.3.3: \( \{ (x, y) \in \mathbb{R} \times \mathbb{R} : x < y \} \) is a binary relation over the set of real numbers.

Now we will consider certain properties of binary relations:

**Definition 0.3.4:** A binary relation \( R \) over a set \( S \) is said to be

- complete if and only if \( \forall x, y \in S \), we have \( (x, y) \in R \) or \( (y, x) \in R \)
- connected if and only if \( \forall x \in S \) and \( y \in S \setminus \{x\} \), we have \( (x, y) \in R \) or \( (y, x) \in R \)
- reflexive if and only if \( \forall x \in S \), we have \( (x, x) \in R \)
- irreflexive if and only if \( \forall x \in S \), we have \( (x, x) \notin R \)
- symmetric if and only if \( \forall x, y \in S \), we have \( (x, y) \in R \Rightarrow (y, x) \in R \)
- transitive if and only if \( \forall x, y, z \in S \), we have \( (x, y) \in R \) and \( (y, z) \in R \Rightarrow (x, z) \in R \)
- asymmetric if and only if \( \forall x, y \in S \), we have \( (x, y) \in R \Rightarrow (y, x) \notin R \)
- antisymmetric if and only if \( \forall x, y \in S \), we have \( (x, y) \in R \) and \( (y, x) \in R \Rightarrow x = y \)

**Definition 0.3.5:** A binary relation \( R \) which is reflexive, symmetric and transitive is called an *equivalence relation*.

**Definition 0.3.6:** A binary relation \( R \) which is complete and transitive is called a *complete pre-order*.

**Definition 0.3.7:** A binary relation \( R \) which is connected, transitive and asymmetric is called an *order relation*.
0.4 FUNCTIONS

Functions, which are also called mappings are a very important type of relation between two sets. We define them as follows:

**Definition 0.4.1:** Let A and B be two non-empty sets. A relation f from A into B is called a function (or mapping) from A into B if every element of A is the first component of a single ordered pair in \( f \subseteq A \times B \).

We denote such a function as \( f: A \rightarrow B \) and we write \( f(a) = b \) whenever \( (a, b) \in f \). We call b the *image* of a under f. Similarly we call a the *inverse image* of b under f.

Here A is called the *domain* of f and B is called the *range* of f.

**Remark 0.4.1:** Given any function \( f: A \rightarrow B \), \( \forall a \in A \), there exists one and only one \( f(a) \in B \).

Let us now define some important concepts about functions. The first one of these is the equality of two functions defined over a same domain and range, which requires that each element in the range has the same image under both functions. The formal definition is as follows:

**Definition 0.4.2:** Two functions \( f: A \rightarrow B \) and \( g: A \rightarrow B \) are said to be *equal* if and only if \( \forall a \in A \) we have \( f(a) = g(a) \).

We had remarked that by the definition of a function \( f: A \rightarrow B \), each element belonging to the domain A has one and only one image. Note that the converse need not to be true: it is not necessary that each element of the range B has an inverse image. But there are functions which satisfy this, which are called onto functions. The formal definition is as follows:
**Definition 0.4.3:** A function \( f: A \rightarrow B \) is said to be *onto* if and only if \( \forall b \in B, \exists a \in A \) such that \( f(a) = b \).

Again note that the definition of a function does not exclude the possibility that each element of the range \( B \) has more than one inverse image. But again there are functions where no element of the range has more than one inverse image. These are called one-to-one functions. The formal definition is as follows:

**Definition 0.4.4:** A function \( f: A \rightarrow B \) is said to be *one-to-one* if and only if \( \forall a_1, a_2 \in A, \) we have \( f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \).

**Definition 0.4.5:** A function \( f: A \rightarrow B \) which is one-to-one and onto is called a *bijection*.

We will finally define what we mean by the composition of two functions.

**Definition 0.4.6:** Given two functions \( f: B \rightarrow C \) and \( g: A \rightarrow B \), we can define their composition \( f \circ g: A \rightarrow C \) as follows: \( \forall a \in A, f \circ g(a) = f(g(a)) \).
EXERCISES

1. Give an example of a necessary condition which is not sufficient.
2. Give an example of a sufficient condition which is not necessary.
3. Explain the difference between the following two sentences:
   “A set is finite if and only if its cardinality is finite”
   “A set is finite if its cardinality is finite”
4. Prove or disprove the following proposition:
   \( \forall n \in \mathbb{N}, \text{ we have } n \geq 0 \) (\( \mathbb{N} \) is the set of natural numbers)
5. Prove or disprove the following proposition:
   \( \forall n \in \mathbb{N}, \text{ we have } n \geq 5 \)
6. Prove or disprove the following proposition:
   \( \exists n \in \mathbb{N}, \text{ such that } n \geq 5 \)
7. Show that if \( R \) is a subset of \( T \) and \( T \) is a subset of \( S \), then \( R \) is a subset of \( S \).
8. Prove that for any two sets \( S \) and \( T \), we have \( S \cap T = S \) if and only if \( S \subset T \)
9. Prove that for any two sets \( S \) and \( T \), we have \( S \cup T = S \) if and only if \( T \subset S \)
10. Prove that for any two sets \( S \) and \( T \), we have \( (S \setminus T) \cup (T \setminus S) = (S \cup T) \setminus (S \cap T) \)
11. Let \( S \) and \( T \) be finite sets. Prove that if \( S \cap T = \emptyset \) then \( |S \cup T| = |S| + |T| \)
12. Let \( S \) and \( T \) be finite sets. Prove that \( |S \cup T| = |S| + |T| - |S \cap T| \)
13. Write the power set of the set \( S = \{1, 2, 3\} \)
14. Given the set \( S = \{1, 2, 3\} \), write explicitly the set \( T \subset 2^S \) which is defined as
   \[ T = \{ X \in 2^S : |X| \geq 2 \} \]
15. Given any two sets \( S \) and \( T \), prove that \( S \subset T \) if and only if \( 2^S \subset 2^T \).
16. Let \( S \) and \( T \) be finite sets. Show that \( |S \times T| = |T \times S| = |S| \cdot |T| \)
17. Consider the binary relation \( R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x + y = 4\} \) over \( \mathbb{R} \). Prove or disprove the following statements:

(i) \( R \) is reflexive
(ii) \( R \) is irreflexive
(iii) \( R \) is symmetric
(iv) \( R \) is antisymmetric
(v) \( R \) is transitive
(vi) \( R \) is complete
(vii) \( R \) is connected
(viii) \( R \) is asymmetric

18. Solve question 17 for the binary relation \( R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x < y\} \) over \( \mathbb{R} \).

19. Solve question 17 for the binary relation \( R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x \leq y\} \) over \( \mathbb{R} \).

20. Solve question 17 for the binary relation \( R = \{(x, y) \in \mathbb{R} \times \mathbb{R} : x = y\} \) over \( \mathbb{R} \).

21. Show that a binary relation is connected if it is complete.

22. Show that a binary relation is reflexive if it is complete.

23. Show that a binary relation which is connected need not be complete.

24. Show that a binary relation which is reflexive need not be complete.

25. Show that a binary relation is connected and reflexive if and only if it is complete.

26. Show that a binary relation is asymmetric if it is transitive and irreflexive.

27. Define a binary relation \( R \) on \( \mathbb{N} \times \mathbb{N} \) as follows:

\[ ((a, b), (c, d)) \in R \text{ if and only if } a + d = b + c \]

Show that \( R \) is an equivalence relation.

28. Consider the sets \( A = \{a, b, c\} \) and \( B = \{d, e, f\} \). Give an example of a function \( f: A \to B \) which is one-to-one but not onto.

29. Consider the sets \( A = \{a, b, c\} \) and \( B = \{d, e, f\} \). Give an example of a function \( f: A \to B \) which is onto but not one-to-one.

30. Let \( A \) and \( B \) be two non-empty finite sets. Prove or disprove the following statement: “A function \( f: A \to B \) is bijection only if \( |A| = |B| \)”.

31. Let \( A \) and \( B \) be two non-empty finite sets. Prove or disprove the following statement: “A function \( f: A \to B \) is bijection if and only if \( |A| = |B| \)”.
32. Consider the set $A = \{a, b, c\}$. Give an example of a function $f: A \to A$.

33. Consider the set $A = \{a, b, c\}$. How many functions $f: A \to A$ exist.

34. Consider the set $A = \{a, b\}$. Give an example of a function $f: 2^A \to 2^A$.

35. Consider the set $A = \{a, b\}$. Give an example of a function $f: 2^A \to 2^A$ satisfying

   $\forall B \in 2^A, f(B) \subseteq B$. Such a function is called a choice function.

36. Consider the set $A = \{a, b\}$. Give an example of a function $f: 2^A \to 2^A$ which is not a choice function.

37. Given the set $A = \{a, b\}$, how many functions $f: 2^A \to 2^A$ do we have?

38. Given the set $A = \{a, b\}$, how many choice functions $f: 2^A \to 2^A$ do we have?

39. Given a set $A$, a choice function $f: 2^A \to 2^A$ is said to be non-empty if $\forall B \in 2^A \setminus \{\emptyset\}$ we have $f(B) \in 2^A \setminus \{\emptyset\}$. How many non-empty choice functions $f: 2^A \to 2^A$ one can define on the set $A = \{a, b\}$?