A CHARACTERIZATION OF SUPERDICTATORIAL DOMAINS FOR STRATEGY-PROOF SOCIAL CHOICE FUNCTIONS

by

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ABSTRACT

A domain D is dictatorial iff there exists no surjective, strategy-proof and non-dictatorial social choice function defined over D. A dictatorial domain D is superdictatorial iff every superdomain of D is also dictatorial. The existence of dictatorial but not superdictatorial domains being known, we show that a dictatorial domain D is superdictatorial iff every alternative is ranked at the top by some ordering in D.

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1. INTRODUCTION

We know since Gibbard (1973) and Satterthwaite (1975) that for a society confronting at least three alternatives, there exists no surjective, strategy-proof and non-dictatorial social choice function defined over the full domain of preference profiles. However, the full domain assumption is not necessary for this impossibility to hold. For instance, Aswal et al. (2003) show that the Gibbard-Satterthwaite impossibility may prevail under severe restrictions of the full domain.

What forms the subject matter of this paper is an observation in the literature on strategy-proof social choice rules under restricted domains: Take any dictatorial domain of preferences D which is dictatorial, i.e., the Gibbard-Satterthwaite impossibility prevails over D. It is possible that D has superdomains which fail to be dictatorial. So we explore the conditions under which dictatorial domains are also superdictatorial. We show that a dictatorial domain D is superdictatorial if and only if every alternative is ranked at the top by some ordering in D.

Section 2 gives the basic notions. Section 3 states and shows our characterization theorem formally.

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1 By the full domain, we mean that every (strict) ordering of alternatives is admissible as an individual preference.
2 Examples of this failure can be found in Barberà et al. (2001) and Ozyurt and Sanver (2006).
3 A dictatorial domain of preferences D is superdictatorial iff every superdomain of D is also dictatorial. We owe the terminology to Bordes and Le Breton (1990) as well as Kelly (1994), who make the same observation/analysis for the impossibility theorem of Arrow (1963). See Ozdemir and Sanver (2006) for a detailed treatment of the matter.
2. BASIC NOTIONS

Taking any integer \( n \geq 2 \), consider a society \( N = \{1, \ldots, n\} \) and a finite set of alternatives \( A \) with \( \#A \geq 3 \). We write \( \mathcal{R} \) for the set of complete, transitive and antisymmetric binary relations over \( A \). Each \( i \in N \) has a preference \( R_i \in \mathcal{R} \) over \( A \) and a preference profile is an \( n \)-tuple \( \mathbf{R} \in \mathcal{R}^N \) of individual preferences. Letting \( D \subseteq \mathcal{R} \) stand for an arbitrary non-empty subdomain of \( \mathcal{R} \), we define a social choice function (SCF) as a mapping \( f: D^N \to A \). A SCF \( f: D^N \to A \) is surjective iff given any \( x \in A \), there exists \( R \in D^N \) such that \( f(R) = x \). A SCF \( f: D^N \to A \) is manipulable by \( i \in N \) at \( R \in D^N \) iff there exists \( R' \in D^N \) with \( R'_j = R_j \) for all \( j \in N \setminus \{i\} \) such that \( f(R') \neq f(R) \) and \( f(R') R_i f(R) \). A SCF \( f: D^N \to A \) is strategy-proof iff \( f \) is manipulable at no \( R \in D^N \), by no \( i \in N \). A SCF \( f: D^N \to A \) is dictatorial iff there exists \( d \in N \) such that \( f(R) = R_d \) at each \( R \in D^N \).

We say that a domain \( D \) is dictatorial iff \( D \) admits no surjective, non-dictatorial and strategy-proof SCF \( f: D^N \to A \).\(^4\) A dictatorial domain \( D \) is superdictatorial iff any \( E \subseteq \mathcal{R} \) with \( E \supseteq D \) is also dictatorial. We qualify \( D \) as regular iff for all \( x \in A \), there exists \( R \in D \) such that \( \text{argmax}_A R = x \).\(^5\)

3. THE CHARACTERIZATION THEOREM

**Theorem 1:** A dictatorial domain \( D \subseteq \mathcal{R} \) is superdictatorial if and only if \( D \) is regular.

**Proof:** Take any dictatorial \( D \subseteq \mathcal{R} \). To see the “if” part let \( D \) be regular and take any \( E \subseteq \mathcal{R} \) with \( E \supseteq D \). Consider any surjective and strategy-proof \( f: E^N \to A \). We will show, through the three lemmata below, that \( f \) is dictatorial.

\(^4\) The Gibbard-Satterthwaite Theorem establishes the dictatoriality of \( \mathcal{R} \).

\(^5\) We say \( \text{argmax}_A R = x \) whenever \( x R y \) for all \( y \in A \).
**Lemma 1:** There exists \( d \in \mathbb{N} \) such that for all \( R \in E^N \) with \( R_i \in D \) \( \forall i \in N \), we have 
\[ f(R) = \text{argmax}_A R_d. \]

**Proof of Lemma 1:** Define \( f_D : D^N \rightarrow A \) as \( f_D(R) = f(R) \) for all \( R \in D^N \). As \( f \) is strategy-proof, \( f_D \) is strategy-proof as well. As \( f_D \) is strategy-proof and \( D \) is regular, \( f_D \) is surjective. Finally, as \( D \) is dictatorial, there exists \( d \in N \) such that for all \( R \in D^N \), we have 
\[ f_D(R) = \text{argmax}_A R_d, \]
which by the construction of \( f_D \), establishes the existence of \( d \in N \) such that for all \( R \in E^N \) with \( R_i \in D \) \( \forall i \in N \), we have 
\[ f(R) = \text{argmax}_A R_d. \]

\( \blacksquare \)

**Lemma 2:** There exists \( d \in N \) such that for all \( R \in E^N \) with \( R_d \in D \), we have 
\[ f(R) = \text{argmax}_A R_d. \]

**Proof of Lemma 2:** Take any \( R \in E^N \). We know by Lemma 1 the existence of \( 1 \in N \) such that whenever \( R_i \in D \) \( \forall i \in N \), we have 
\[ f(R) = \text{argmax}_A R_1. \]
We claim that whenever \( R_1 \in D \) and \( \# \{ i \in N \setminus \{1\} : R_i \in E \setminus D \} = k \) for some \( k \in \{1, \ldots, n-1\} \), we have 
\[ f(R) = \text{argmax}_A R_k. \]
We prove our claim by induction. So we first show our claim for \( k=1 \). Hence, we will show that if \( R_1 \in D \), \( R_2 \in E \setminus D \) for some \( 2 \in N \setminus \{1\} \) and \( R_i \in D \) for all \( i \in N \setminus \{1, 2\} \), then 
\[ f(R) = \text{argmax}_A R_1. \]
To see this, take any such \( R \in E^N \), let \( \text{argmax}_A R_1 = a \) and suppose \( f(R) = b \) for some \( b \in A \setminus \{a\} \). If a \( R_2 \) \( b \) then agent 2 can manipulate \( f \) at \( R \) by pretending any \( R \in D \). Now, consider the case where \( b \neq R_2 \). As \( D \) is regular, there exists \( R \in D \) with \( \text{argmax}_A R = b \). Take any \( R' \in E^N \) such that \( R_i = R_i \) for all \( i \in N \setminus \{2\} \) and \( R_2' = R \). By Lemma 1, we have \( f(R') = a \) and agent 2 can manipulate \( f \) at \( R' \) by pretending \( R_2 \).

We proceed with the induction step and show that if our claim holds for some \( k \in \{1, \ldots, n-2\} \), then it holds for \( k+1 \) as well. So suppose we have \( f(R) = \text{argmax}_A R_i \) whenever \( \# \{ i \in N \setminus \{1\} : R_i \in E \setminus D \} = k-1 \). We will show that \( f(R) = \text{argmax}_A R_i \) whenever \( \# \{ i \in N \setminus \{1\} : R_i \in E \setminus D \} = k \) as well. To see this, take any such \( R \in E^N \), let \( \text{argmax}_A R_i = a \) and assume without loss of generality that \( \{ j \in N : 2 \leq j \leq k+1 \} \) is the set of agents whose preferences belong to \( E \setminus D \). Suppose, for a contradiction, that \( f(R) = b \) for some \( b \in A \setminus \{a\} \). If a \( R_2 \) \( b \) then agent 2 can manipulate \( f \) at \( R \) by
pretending any \( R \in D \). Now, consider the case where \( b R_2 a \). As \( D \) is regular, there exists \( R \in D \) with \( \text{argmax}_A R = b \). Take any \( R' \in E^N \) such that \( R'_i = R_i \) for all \( i \in N \backslash \{2\} \) and \( R_2' = R \). We have \( f(R') = a \) and agent 2 can manipulate \( f \) at \( R' \) by pretending \( R_2 \).

**Lemma 3:** There exists \( d \in N \) such that for all \( R \in E^N \) with \( R_d \in E \backslash D \), we have \( f(R) = \text{argmax}_A R_d \).

**Proof of Lemma 3:** Take any \( R \in E^N \). We know by Lemma 2 the existence of \( 1 \in N \) such that whenever \( R_d \in D \), we have \( f(R) = \text{argmax}_A R_d \). Now let \( R_1 \in E \backslash D \) and \( a = \text{argmax}_A R_1 \). Suppose for a contradiction that \( f(R) = b \) for some \( b \in A \backslash \{a\} \). However, by Lemma 2, \( f(R') = a \) at \( R' \in E^N \) with \( R_j' = R_j \) for all \( j \in N \backslash \{1\} \) and \( R_1' \in D \) with \( \text{argmax}_A R_1' = a \). Hence agent 1 can manipulate \( R \) by pretending \( R_1' \in D \), contradicting that \( f \) is strategy-proof.

Lemma 2 and Lemma 3 establish the “if” part of Theorem 1.

To see the “only if” part, suppose \( D \) is not regular. So there exists \( x \in A \) such that \( \text{argmax}_A R = x \) for no \( R \in D \). Take some \( R^* \in \mathbb{R} \backslash D \) such that \( \text{argmax}_A R = x \). Let \( y \in A \) be such that \( y R^* z \) for all \( z \in A \backslash \{x\} \). We define a SCF \( f : [D \cup \{R^*\}]^N \rightarrow A \) over the domain \( D \cup \{R^*\} \) such that for all \( R \in [D \cup \{R^*\}]^N \), we have

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\begin{align*}
\text{argmax}_A R_1 & \quad \text{when } R_1 \in D \\
\text{argmax}_{\{x,y\}} R_2 & \quad \text{when } R_1 = R^*
\end{align*}
\]

Checking that \( f \) is surjective, non-dictatorial and strategy-proof completes the proof of Theorem 1.\(^6\) ■

\(^6\) \( f \) is used by Barberà et al. (2001) in a more structured framework, with a similar purpose.
REFERENCES


